Near Optimum Sampling for Single Tone Frequency Estimation

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Abstract—Work on single tone frequency estimation has focused on uniformly sampled data. However, it has been shown that, for a given number of samples, more information on the frequency of a signal can be gained by non-uniform sampling schemes. Unfortunately, an optimum sampling pattern (that, for example, minimizes the Cramér-Rao bound) does not automatically have a fast and simple algorithm for frequency estimation associated with it. For application in an interferometric measurement system, an algorithm is needed that gathers as much information as possible from a low number of samples, while at the same time keeping the computational effort sufficiently low to process millions of time series in a few seconds. One simple approximation to the optimum pattern can be obtained by 1) uniformly sampling blocks of data, 2) estimating phase and frequency in each of these and 3) exploiting these intermediate results in the final estimation. An approach to do so is investigated in detail. Results are compared to the Cramér-Rao bound (CRB), and it is shown that the algorithm almost reaches this limit on the variance of unbiased estimators, at a computational complexity lower than that of a typical FFT-based approach. For M=32 samples and a signal-to-noise ratio of 10, the standard deviation of the frequency estimate is lower by more than 50% compared to uniform sampling. In addition, the algorithm can easily be applied to poorly characterized systems, e.g. systems for which the noise is not known exactly.

Index Terms—Sampling, Frequency Estimation, Optimization, Experimental Design

I. INTRODUCTION

FAST and accurate frequency estimation for a noisy sinusoid is required in many applications today, ranging from acoustic [1] and radar [2] signal processing to optical metrology [3].

There are many techniques for the analysis of periodic signals, both stationary and time-varying. Methods include the (windowed) FFT [4]–[7], autocorrelation based signal subspace techniques (Pisarenko's method, MUSIC, ESPRIT [8]–[10]), non-linear optimization techniques (iterative, in both time [11] and frequency domain [12]) and filter based techniques [13]. A large number of papers and many textbooks cover these methods, including [14]–[16]. In most cases, uniform sampling is used as this is the easiest way to acquire data, for example if an electrical signal is recorded with an analog-to-digital-converter. Such systems have a constant sampling rate and acquire the signal for a given time period. Continuous broadband signals must be band-limited by the

Nyquist frequency to allow for exact reconstruction from the samples and avoid aliasing [17]. In case of single tone frequency estimation, this aliasing is not necessarily a problem: It causes ambiguity, but the frequency estimates are still accurate, and the actual frequency can be determined if prior knowledge to resolve the ambiguity is available. The accuracy of such a system largely depends on the sampling time, i.e. on the number of samples.

In some cases, however, the situation is different:

- For some applications (e.g. antialiasing in computer graphics [18]), better (in the case of computer graphics visually more pleasing) results can be obtained with random sampling instead of uniform sampling. This is not discussed here.
- The sampling operation itself and the signal processing may be expensive, and therefore a low number of samples (but not necessarily uniform or close to each other) might be desirable. For instance, in frequency scanning interferometry, and more generally in all applications where the sinusoidal signal is explicitly sampled by choosing specific sampling points, there is a cost associated with the number of samples rather than with their spacing.

In the latter cases, sampling can be accelerated and costs reduced by carefully choosing the optimum sampling points. An optimum sampling scheme for a limited sampling range and sampling time with an arbitrary distribution of the sampling time across the samples has been introduced in [19]. While the proposed sampling design is optimal given the above constraints, it does not, by itself, suggest an algorithm to efficiently estimate the frequency from the resulting data.

This article proposes and analyzes a sampling strategy that yields results very close to the theoretical optimum while at the same time allowing for a fast, robust and intuitive algorithm. In section II, the signal model is specified and the optimization criteria are motivated and described. The algorithm is derived in section III. Section IV offers bounds on the accuracy of the algorithm, and a comparison with alternative sampling strategies. The results are verified with simulations. Section V briefly discusses several possible extensions to the algorithm and section VI summarizes the implementation of the algorithm and its key properties.

II. SIGNAL MODEL AND OPTIMIZATION CRITERIA

We would like to accurately estimate, from as few samples as possible, the frequency of a noisy single tone

$$I(t) = A \cdot \cos(\omega \cdot t_n + \varphi) + C + \epsilon_n,$$
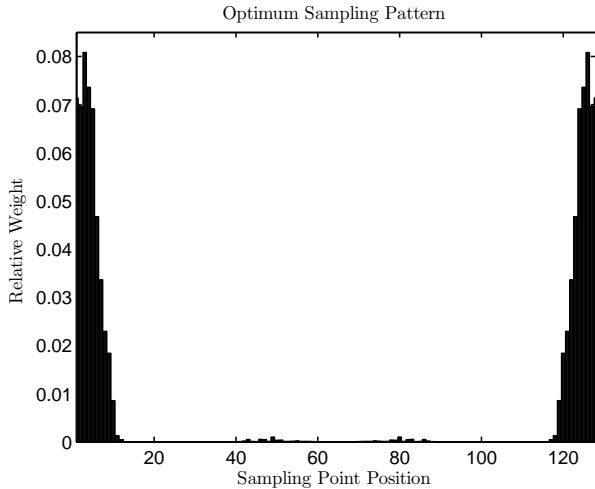$$t_{min} < t_n < t_{max}, \ n = 1, ..., N \tag{1}$$

Fig. 1. Optimum sampling pattern for frequency estimation for $N = 128$ equidistant samples, probability of outliers $P_e = 1\%$. The weights indicate the effort that should be devoted to the acquisition of a measurement at each sampling point. The total sampling effort is equivalent to acquiring 32 samples with SNR 10.

with unknown amplitude A, offset C, frequency ω and ϵ_n independently and identically distributed Gaussian noise with zero mean and variance σ^2. The sampling points t_n should be chosen such that the most accurate estimate for ω can be obtained, and they need not be spaced equidistantly in time. "Most accurate" is defined as maximizing the Fisher information (which is a measure of local curvature) while enforcing a minimum distance to secondary minima: The frequency estimation might become ambiguous in the presence of noise if the difference between the true signal and a signal of a different frequency becomes too small. This is illustrated by Fig. 3 [19]. The signal at the sampling points can be almost identical for different frequencies (especially since the offset and amplitude are also unknown, for appropriate settings the); for samples mainly at the borders of the sampling range, the positions of the alternative frequencies are well-known and can be taken into account in optimization.

It has been shown in [20] that the sampling points near the boundary of the permissible sampling range are most important, and it has been shown in [19] that — taking ambiguity issues into account — using several sampling points mainly at the borders yields optimum results. In that case, the optimum sampling pattern for an application depends on the assumed signal-to-noise ratio, as this ratio determines the ambiguity threshold that is required to keep the probability of outliers (cases where a secondary minimum as shown in Fig. 3 is higher than the primary, correct solution) below a given level.

The optimum sample distribution according to [19] for 128 equispaced sampling points is indicated in Fig. 1. In practice, the "relative weights" translate to "sampling effort", where it is assumed that the variance of a measured value is cut in half if the effort at that point is doubled. In the case of frequency scanning interferometry, sampling effort is measured by the acquisition time at a given point.

Fig. 1 demonstrates that the optimum weight distribution focuses on the sampling points at the borders of the considered range, and only few samples with very low weights can be found in-between.

Arbitrary weights are hard to implement in practice, though: For instance, while there may be a theoretical benefit in varying the time dedicated to sampling at different frequencies in frequency scanning interferometry experiments, with a camera it is much easier to use the same camera exposure time for all samples, as close as possible to the limit dictated by the full well capacity of the sensor [21].

An obvious approximation to the pattern in Fig. 1 is given by simply using two blocks of equally weighted samples at the borders of the range. The impact of the distance and the size of these blocks is analyzed in detail in the next section, and the results are compared to both uniform sampling and the optimum sampling pattern.

III. DERIVATION OF A FAST ALGORITHM

For the reasons given above, an algorithm for evaluating experimental data that has been sampled in multiple blocks is derived and its properties are analyzed. For a sampling pattern that consists of multiple blocks of equally spaced and uniformly weighted samples, there is a straightforward procedure: First, the frequency and phase of the signal are determined for each block individually, and then the results are used to initialize a final estimate based on all observations. Fortunately, there is a very simple and highly accurate way of combining information from multiple blocks, as detailed below. The key to the following algorithm is the simple observation that, visually speaking, frequency is the slope of the phase. Consider the signal from eq. 1 sampled in two blocks centered at t_1 and t_2. For each block, ω and φ can be estimated separately. Then, the following relationship holds as illustrated by Fig. 2:

$$\varphi_1 + (t_2 - t_1) \cdot \omega = \varphi_2 + 2\pi k, \ k \in \mathbb{N} \tag{2}$$

As k is unknown, there is no unique solution for ω.

As a first guess for the frequency, the mean value of the frequency estimates from each of the blocks can be used (strictly speaking, only a frequency estimate from one block is required, but multiple blocks are needed for the phase estimation anyway):

$$\hat{\omega}_{init} = \frac{\hat{\omega}_1 + \ldots + \hat{\omega}_N}{N} \tag{3}$$

k is then chosen such that

$$\Delta = \hat{\varphi}_1 - \hat{\omega}_{init} t_1 - (\hat{\varphi}_2 - \hat{\omega}_{init} t_2) - 2\pi k, \ k \in \mathbb{N} \tag{4}$$

is minimized:

$$\hat{k}_{opt} = \arg \min_k (\hat{\varphi}_1 + \hat{\omega}_{init} \cdot (t_2 - t_1) - \hat{\varphi}_2 + 2\pi k) \tag{5}$$

Ambiguities are resolved correctly as long as the combined error caused by frequency and phase estimation errors as well as unknown sampling jitter does not exceed π.

Next, an improved frequency estimate can be computed. Its accuracy depends on the accuracy of the phase estimation only.

$$\hat{\omega}_{new} = \frac{\hat{\varphi}_2 - \hat{\varphi}_1 + 2\pi \hat{k}_{opt}}{t_2 - t_1} \tag{6}$$
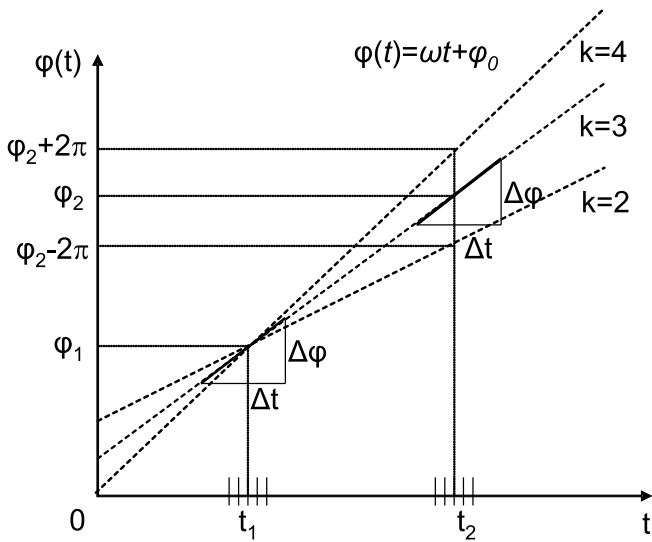
Fig. 2. Relationship between phase and frequency: Phase estimates from multiple blocks of data can be combined to obtain a more accurate frequency estimate. The phase values of the blocks define a grid of possible phase slopes (=frequencies), the correct one is chosen based on the frequency estimates from the individual blocks.
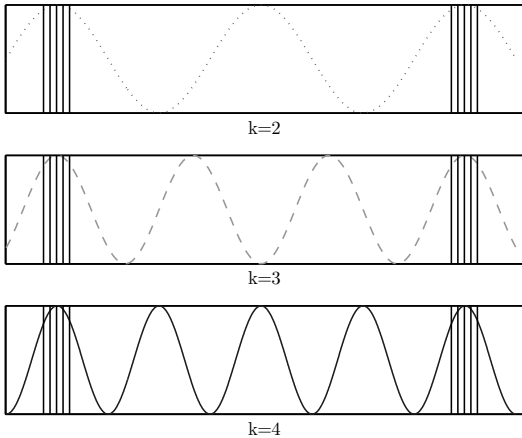

Fig. 3. Possible frequencies based on the phase estimates and the frequency estimate from the individual blocks. The signal at the sampling positions is very similar for all three cases depicted here, which can lead to a wrong k being chosen.

The results of the phase estimation define a "ladder" of frequencies that are more or less compatible with the observations; the initial rough frequency estimate is used to find k and thus identify the "right step on this ladder". Provided that k, the number of wavelengths between the sampling blocks, is correctly found, the accuracy of the final result depends only on the accuracy of the phase estimates, the accuracy of the block distance estimate (which might be influenced by sampling jitter) and the absolute distance of the blocks (a larger distance increases accuracy). The accuracy of the initial frequency estimate and the distance between the blocks determine the probability of outliers P_e, i.e. situations in which the estimate k is wrong (a smaller block distance reduces this probability).

Processing is very fast for this algorithm due to two factors:

1) The number of samples $2 \cdot M$ is much lower than N in case of uniform sampling, and all computationally expensive steps can be done per block individually, requiring only a very low number of samples and therefore little memory in every step.
2) The computational complexity is not higher than that of any other frequency estimation algorithm applicable to a low number of samples, i.e. $O(N \log N)$ in case of a typical FFT based implementation. For two blocks with $M < N/2$ samples, the computational effort ($2 \cdot M \log M$) is lower than for processing of the uniformly sampled data ($N \log N$). Combining the results from multiple blocks needs a fixed low effort only.

In case of more than two blocks, the approach above can be applied iteratively, starting out with the two blocks with the smallest distance, and then consecutively choosing pairs of blocks with increasing distance, but using $\hat{\omega}_{new}$ obtained from the previous two blocks instead of $\hat{\omega}_{init}$. This procedure can be repeated until the two blocks with maximum distance are used, and therefore this leads to the same accuracy as if only the blocks with the largest distance were used, but with a lower probability of outliers (incorrect k).

If a certain number of outliers must not be exceeded, there are three ways to reach this goal, with different drawbacks:

- Increasing the number of sampling points per block increases measurement and processing time.
- Increasing the number of blocks also increases measurement and processing time.
- Reducing the distance between the blocks reduces accuracy.

IV. COMPARISON OF THE ALGORITHM TO THE THEORETICAL BOUND FOR UNIFORM AND FOR OPTIMUM SAMPLING

The performance of the proposed algorithm is compared to the theoretical lower bound (CRB) of the variance for both the theoretically optimum sampling pattern and the uniform sampling pattern. Data acquisition time, processing time and accuracy as well as robustness to outliers are discussed. For that purpose, an approximation for the probability of outliers P_e and for the accuracy of the algorithm given in section III is derived.

For the frequency estimation accuracy of the individual blocks an (approximate) lower bound is derived in [5]. This bound applies to complex signals only, in the real-valued case the bound depends on the true frequency and phase of the signal. In addition, the bound in [5] does not take an unknown signal offset into account. Asymptotically though, the variance of the real valued case with unknown offset approaches twice the variance of the complex valued case (which is intuitively clear as only half the number of independent measurements are assumed to be available). This is briefly shown in the appendix.

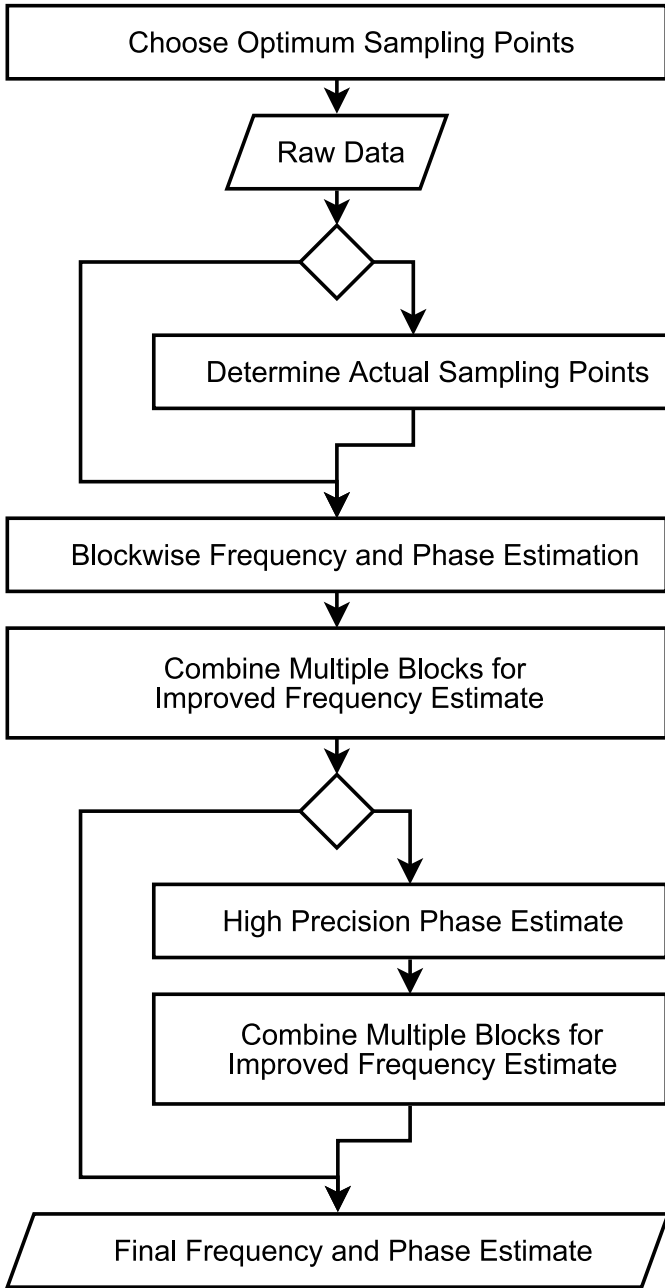The relative standard deviation is then given by the square

Fig. 4. Block diagram of the proposed algorithm for frequency estimation. Once the optimum sampling pattern for the application has been chosen, the raw data is acquired. Optionally the actual sampling positions might be determined for use in the algorithm. Then a frequency and phase estimate for each block of data is obtained, and a new frequency estimate is computed using the algorithm described in this paper. Optionally, a high precision phase estimate can be performed on the basis of the new frequency estimate, and can then be used for a new frequency estimate. Finally, the resulting frequency and phase estimates are returned. There are multiple extensions possible, including amplitude estimation, iterative approaches and using prior knowledge, but these are outside the scope of this paper.

root of this approximate variance divided by π,

$$s_{\hat{\omega}} \geq \frac{2\sqrt{6}}{\pi} \cdot \frac{\sigma}{A} \cdot \frac{1}{M\sqrt{M\left(1 - \frac{1}{M^2}\right)}} \qquad (7)$$

Fig. 5 (top) shows the lower bound on the standard deviation as a function of true frequency and phase when taking the
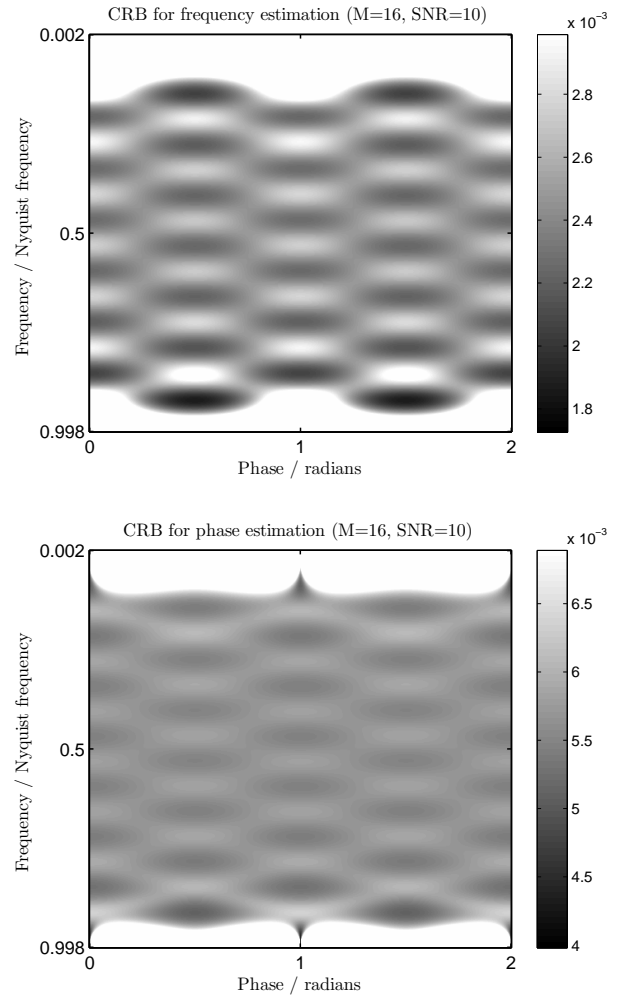

Fig. 5. Theoretical lower limit on the relative standard deviation of a frequency (top) and a phase (bottom) estimate for a single noisy tone (eq. 1) using 16 equispaced samples and a SNR of 10, based on numerical evaluation of the CRB. The plot underlines the fact that estimation becomes unreliable for frequencies close to zero or the Nyquist frequency, where the color scale has been truncated.

unknown offset and the real-valuedness of the signal model into account.

For the phase estimation from a block of samples with known frequency or for the phase in the center of a block of samples with unknown frequency, using the same approximations as above, one obtains [5] a relative standard deviation of

$$s_{\hat{\varphi}} \geq \frac{\sqrt{2}}{2\pi} \cdot \frac{\sigma}{A} \cdot \frac{1}{\sqrt{M}} \qquad (8)$$

Again, eq. 8 does not take the unknown offset and the real signal model into account. The CRB can be computed exactly for the phase estimation, with an approach similar to the one for the frequency estimation, see Fig. 5 (bottom).

Returning to the algorithm described in section III, the probability of outliers (incorrect $\hat{k}_{opt}$ in eq. 5) depends on the frequency and phase estimation accuracy as well as on the inter-block distance. In the following, we assume there are two blocks with M uniform samples each, and a total range (from one edge of one block to the other edge of the other block) of
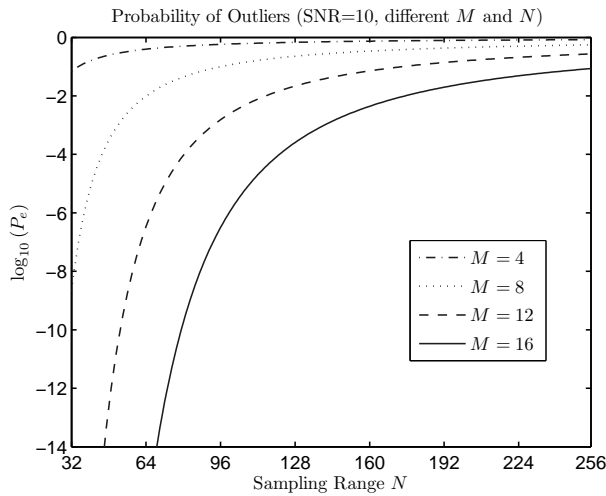
Fig. 6. Probability of outliers P_e as a function of sampling range N and block size M, logarithmic scale.

N (uniform) samples. Then we can compute an approximate variance for Δ defined in eq. 4, assuming independence of the frequency and phase estimates:

$$\text{var}(\Delta) = \sigma^2 \approx 2 \cdot \left(2\pi s_{\hat{\phi}}\right)^2 + \left((N - M) \cdot \pi s_{\hat{\omega}}\right)^2 \quad (9)$$

Equation 9 is only an approximation, though; the real estimation accuracy depends on the true signal frequency and phase and on the algorithms used for frequency and phase estimation. In the optimum case (if the true phase and frequency lead to a minimum in the variance of the phase and frequency estimates), the variance is roughly half the one given above; in the worst case (i.e. when the signal frequency is close to the Nyquist frequency) it can be infinite. In addition, the phase and frequency estimates are not independent of each other, hence the above approach that assumes uncorrelated data is not exact. For frequencies far from zero or the Nyquist frequency (cf. Fig. 5, the exact range depends on the SNR and the number of samples per block) the estimate above is good enough to show some general relations. If one assumes that the distribution of the parameter estimates is approximately Gaussian (which is a good approximation in case of low noise; for high noise the algorithm is not applicable as then a combined analysis of all sampled blocks instead of an analysis of the individual blocks is much better), one can easily compute the probability of outliers P_e: For Δ, a Gaussian distribution with zero mean and variance according to eq. 9 can be assumed. If the absolute value of Δ is larger than π, the phase coupling procedure fails. Thus the probability of outliers is approximated by

$$P_e = P(|\Delta| > \pi) = \text{erfc}\left(\frac{\pi}{\sqrt{2}\sigma}\right) \quad (10)$$

This does not take into account outliers that are caused directly by the M-point frequency estimation, but if the SNR is high enough for the coupling of blocks to work, the probability of outliers occurring in the M-point frequency estimation step is negligible.

The standard deviation as given above is directly proportional to the noise level. For a given number of samples per block and a given SNR, one can compute the maximum (and therefore optimum) block distance N for a previously specified probability of outliers P_e as demonstrated in Fig. 6.

This strategy is easy to implement even if in a practical application analytical treatment becomes difficult: One can simply implement the algorithm and look at a histogram of the phase differences Δ across all pixels in the image. It is then obvious when the algorithm fails (i.e. if the distribution becomes too broad) and very simple to adjust the parameters block size M and block distance $N - M$ empirically such that the desired performance and error probability for a given problem is reached. This strategy can therefore be applied even when the noise is correlated, multiplicative or a simple closed form solution does not exist for other reasons.

The accuracy of the result (disregarding outliers) is given by the accuracy of the phase estimate and block distance only.

$$s_{\omega_{new}} = \sqrt{2} \cdot \frac{2\pi s_\phi}{\pi(N - M)} = \frac{2}{\pi} \cdot \frac{\sigma}{A} \cdot \frac{1}{(N - M) \cdot \sqrt{M}} \quad (11)$$

A lower bound for the error based on the CRB can be computed as shown in Fig. 7.

For the values used in Fig. 7, the root mean squared value of the theoretical limit based on the CRB for the relative accuracy of the frequency estimation in the center frequency range from 0.125 to 0.875 is $1.39 \cdot 10^{-4}$. A numerical estimation on simulated data (using a linear least squares estimator for the phase) yields a standard deviation of approximately $1.40 \cdot 10^{-4}$, which shows that this accuracy can be reached in practice. Both of these values are very close to the approximation in equation 11, which yields $1.42 \cdot 10^{-4}$. Going back to the theoretically optimum sampling pattern as described in [19], the following results are obtained: For $P_e = 2.5 \cdot 10^{-4}$ (approximately the same theoretical probability of outliers as in the case of two blocks with 16 samples each and a total range of 128 samples, according to Fig. 6), a theoretical relative accuracy of $1.35 \cdot 10^{-4}$ is reached. This comparison might be unfair as we have not shown that there is an algorithm that can actually reach such a low probability of outliers, but the probability of outliers of very simple implementations can easily be shown to be far below 1%. With $P_e = 1\%$, as assumed for Fig. 1, the theoretical accuracy improves only slightly to about $1.34 \cdot 10^{-4}$. In contrast, using the same number of samples ($2M = 32$) distributed uniformly across the measurement range N, the relative standard deviation is $2.15 \cdot 10^{-4}$ (in this case excluding the values at the border frequencies relative to the new Nyquist frequency, otherwise the results would be even worse). This shows that the accuracy of the procedure described here is very close to the theoretical limit (to about 3% in this case): Even if arbitrary sampling weights are allowed, there cannot be a significantly better frequency estimation as long as the constraints on sampling range and sampling effort are kept.

A more systematic comparison of various possible sampling strategies yields the results shown in Fig. 8. The following four cases are compared:

1) Uniform sampling with a fixed number of samples $2M$ over a fixed range of samples M:
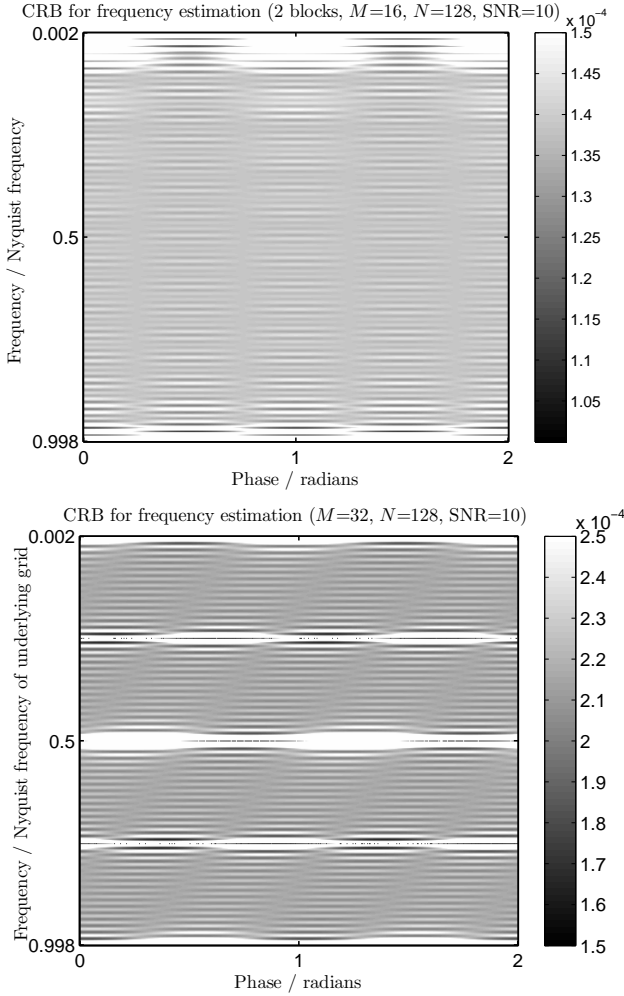
$$t_i = i, \ 1 \le i \le 2M$$

Fig. 7. Lower bound on the standard deviation of the frequency estimate from two blocks with 16 samples each from a range of 128 samples (top); for comparison the same bound is shown for the same total number of samples (32), but now uniformly spaced over the same range of 128 samples (bottom). The areas of zero and Nyquist frequency are clearly visible; and even in between the accuracy is more than 30% lower. The scales for both graphs are different, the scale is truncated in the white areas.

2) Uniform sampling with a number of samples N, same sampling distance as above, N increasing. This would require significantly longer measurements:

$$t_i = i, \ 1 \leq i \leq N \geq 2M$$

3) Uniform sampling with $2M$ samples, but increasing distance of the samples such that the total range is identical to the case with N samples.[1] In practice this would cause ambiguity issues as the Nyquist frequency decreases.

$$t_i = \frac{i}{M} \cdot N, \ 1 \leq i \leq M$$

4) Sampling in two blocks with M samples each, the block distance increasing with N such that the total range is

[1]In addition, we investigated the case of N uniform samples, with lower measurement effort for each, such that the sum of relative weights amounts to $2M < N$. However the results are so similar to case 3) that they have been omitted in the graph of Fig. 8.
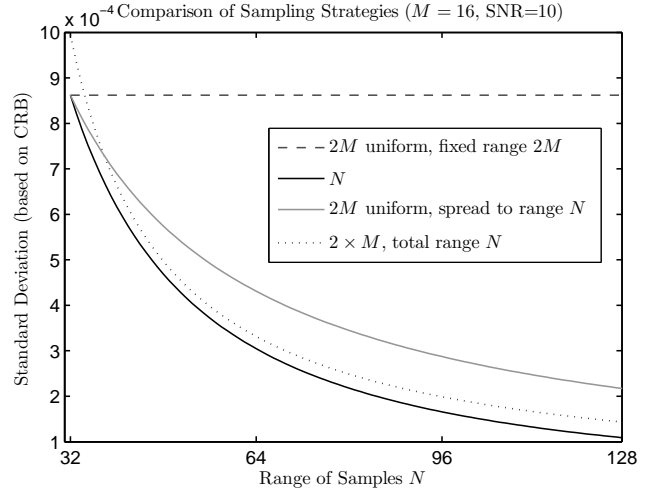

Fig. 8. Comparison of different sampling strategies: Three different ways for uniform sampling (fixed number of samples and distance; fixed number of samples and increasing distance; increasing number of samples) and sampling in two blocks with increasing distance are compared. The sampling strategy depicted with a black line uses $32, \ldots, 128$ samples, whereas all other strategies require 32 samples only. For a range of $N = 128$ and $2M = 32$ samples, the two-block strategy proposed here has a standard deviation which is about 34% lower than that of uniform sampling with the same number of samples and measurement range, at a slightly reduced computational cost and with a larger unambiguous range.

N.

$$t_i = \begin{cases} i, & \text{for } 1 \leq i \leq M \\ N - M + i, & \text{for } M + 1 \leq i \leq 2M \end{cases}$$

This is the main strategy proposed in this paper.

As the results are proportional to the noise level for sufficiently small noise, an SNR of 10:1 was chosen with little loss of generality. The results show that the proposed algorithm has a very good theoretical accuracy if there is a sufficiently large block distance (as long as the upper limit on the inter-block distance dictated by the acceptable probability of outliers is not exceeded) Performance is necessarily worse than using $N >> 2M$ samples. A more detailed comparison between strategy 2) and 4) is offered in Fig. 9.

Sampling with 2×16 samples instead of 1×64 samples decreases measurement time by 50% and processing time (if an FFT based algorithm is used in both cases) by 75%, at the cost of a reduction in accuracy of less than 10%. Even at a quarter of the sampling time, the relative standard deviation increases by only about 32% instead of the 100% expected from the noise.

V. EXTENSIONS

A. Known sampling jitter

Phase estimation still works very well even if the samples are not equally spaced. If the jitter is large, it calls for more sophisticated algorithms for frequency estimation. If the jitter is not too large, it can simply be ignored in the frequency estimation step: For moderate block distances a sub-optimal frequency estimation does not lead to many outliers, and therefore the final result is still close to the theoretical limit.
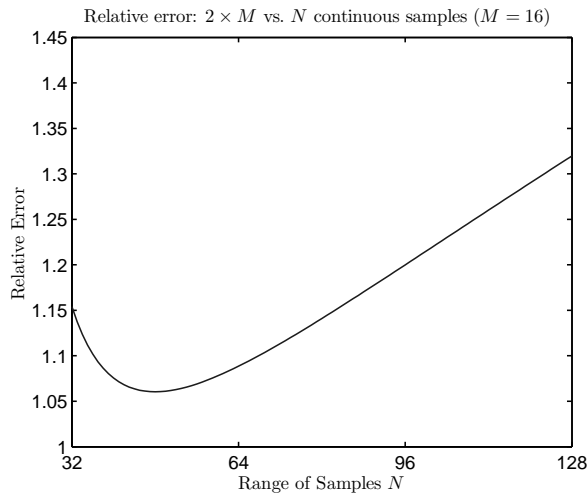
Fig. 9. Relative error of dual block method ($2M = 32$ samples in total, but increasing block distance $N - 2M$) compared to single block method (increasing range and number of samples N and therefore increasing total sampling time).

B. Multiple blocks of data

The algorithm can easily be extended to more than two blocks of data. As described in section III, this can be done by using two blocks of data at a time, starting with the blocks with the smallest distance, and then looking at increasing block distances. In this case the probability of outliers decreases, and the accuracy is determined by the largest available distance. Alternatively, a (weighted) least squares estimate could be obtained from all phase values simultaneously (which is especially relevant if for some reason the outer blocks do not offer good signal quality), but this is only applicable if the maximum distance of the blocks is small enough to avoid outliers.

C. Tracking frequencies

The proposed algorithm can be modified for tracking quickly changing frequencies: In that case one can use the frequency and phase estimate from short blocks, and adjust the block distance used for the final frequency estimate depending on the input data. This can be done for example using a Kalman filter or a simple heuristic approach that increases the distance when the correction based on the phase estimate is small, and decreases it when the correction is larger. This is useful if the tracking is based on blockwise information, e.g. blocks of pilot symbols embedded in a signal [22].

D. Using prior knowledge on the signal phase

If prior knowledge on the signal phase is available for some t, a method for a more accurate frequency estimation can be derived. This can be applied in optical metrology, for example. For a smooth and continuous surface, the result of using this approach in a multiple wavelength interferometry system is identical to that obtained with spatial unwrapping, at a much lower computational cost. In addition to that, the phase estimation can also be used to obtain highly accurate

measurement results for surfaces that are not continuous as long as the surface properties and therefore the signal phase ϕ_0 remains the same.

For this approach, first the frequency ω is determined as accurately as possible with the algorithm described above. Next a single phase value is considered. This phase value can be obtained from only one block or from the whole set of samples. Once again the signal is given by eq. 1. Then in this special case we use our prior knowledge on the phase ϕ_0 and obtain

$$2\pi k = \varphi_0 - (\omega \cdot t + \varphi), \; k \in \mathbb{N}. \tag{12}$$

Again, there is no unique solution and therefore k is chosen such that the difference is minimized:

$$\hat{k}_{opt} = \arg\min_k(\hat{\varphi}_0 - (\hat{\omega} \cdot t + \hat{\varphi}) + 2\pi k) \tag{13}$$

Then an improved frequency estimate can be computed (again, the accuracy depends only on the phase estimate):

$$\hat{w}_{new} = \frac{\hat{\varphi} - \hat{\varphi}_0 + 2\pi\hat{k}_{opt}}{t} \tag{14}$$

This is very similar to the derivation above, but t is usually much larger than $t_2 - t_1$, and therefore the "ladder" of frequencies is very fine and the results are more accurate; but the probability of outliers increases.

If there are outliers, it is usually difficult to change the distance of the blocks (in case of optical metrology this distance is given by the laser frequency). This issue can only be resolved with a sufficiently accurate initial frequency estimate, which can be obtained by the algorithm described earlier, or by using prior knowledge on spatial relationships (e.g. smoothness constraints) to correct incorrect choices of k.

VI. Summary and Conclusion

An efficient sampling scheme and algorithm for single tone frequency estimation has been presented. An implementation of the proposed algorithm consists of the following steps:

- Choice of sampling points (e.g. using two blocks of samples with maximum feasible distance for a desired probability of outliers)
- Frequency estimation for one or more blocks [23]
- Phase estimation for each block [23]
- Determination of the actual sampling points (optional, if available one can take sampling jitter into account)
- Improved frequency estimate by phase coupling
- Absolute phase estimation using prior knowledge (optional, if knowledge is available)

For a practical implementation, the block size and block distance have to be adjusted to reach the desired accuracy and probability of outliers of the frequency estimation.

The proposed algorithm has three key advantages:

- First of all, it is very fast. Processing time depends mainly on the algorithms used for phase and frequency estimation for the individual blocks, the rest of the algorithm takes much less than a second on a current PC. If fast approaches are used for phase and frequency estimation, a total processing time of less than 10s for

1 million frequency estimates using $M = 32$ frames can be achieved in Matlab on an Intel Core 2 Duo E6600 processor. Processing with dedicated hardware or more optimized software is expected to be significantly faster. Computational complexity is usually lower than that of uniform sampling, given the same number of sampling points. In particular, the algorithm is faster than taking the FFT on a single block of M uniformly sampled data points.

- The algorithm is also highly accurate: The performance of this algorithm by far exceeds that of uniform sampling with the same number of samples and gets very close to the theoretical optimum sampling scheme and theoretically best frequency estimation. On the one hand, the sampling pattern is close to the theoretically optimum sampling pattern, and on the other hand, the algorithm almost reaches the CRB (to less than 1% for an SNR better than 1) for this sampling pattern. Altogether, the result is within 3% of the theoretical limit on the accuracy for the theoretically optimum sampling pattern, i.e. any possible improvements are known to be very limited.

- The algorithm is highly flexible: It can easily be extended to take known sampling jitter or multiple sampling blocks into account, without extra computational effort. In addition, one can easily apply the method even if the noise is correlated or unknown by optimizing a histogram of phase differences as computed from eq. 4. The proposed method can therefore be generalized to a wide variety of applications.

APPENDIX
CRB FOR REAL-VALUED SIGNAL

The Cram'er-Rao bound for unbiased estimators is given by

$$\text{Var}_\theta \geq \frac{1}{I_\theta} = \frac{1}{E_\theta\left[\left(\frac{\partial}{\partial\theta}\log p_\theta(Y)\right)^2\right]} \quad (15)$$

For a general signal model $y(t,\theta)$ with a vector parameter θ and multivariate Gaussian noise n with covariance matrix Σ

$$x = y(t,\theta) + n \quad (16)$$

this leads to the following equation for elements of the Fisher information matrix I_θ [24]:

$$I_{j,k}(y;t,\theta) = \left(\frac{\partial y}{\partial\theta_j}\right)^T \Sigma^{-1} \left(\frac{\partial y}{\partial\theta_j}\right) \quad (17)$$

Applying this result to the real-valued sinusoidal signal model given by eq. 1 yields the 4×4 Fisher information matrix I. The diagonal elements of the inverse of this matrix represent the lower bound on the variance for unbiased estimators of the frequency, phase, amplitude and offset. The corresponding element for the frequency is visualized in Fig. 5.

The exact results for the real-valued case require more space than is available here and are not very instructive. For practical applications a simplified result is often sufficient. For frequency estimation with known phase, amplitude and offset, only a single element of the Fisher information matrix has to be inverted. If additionally the signal is sampled uniformly

and the noise is independent and identically distributed, one obtains:

$$\begin{aligned} I_{11} &= \frac{A^2}{\sigma^2} \sum_{n=0}^{N-1} {t_n}^2 \sin^2(\omega t_n - \phi) \\ &= \frac{A^2}{2\sigma^2}\left(\sum_{n=0}^{N-1} {t_n}^2 - \sum_{n=0}^{N-1} {t_n}^2 \cos(2\omega t_n - 2\phi)\right) \end{aligned} \quad (18)$$

This looks very different from the results given by [5], but one can easily show that the results are closely related: The second sum is much smaller than the first one for almost all combinations of sampling points and frequencies, and the more samples the bigger the difference gets. In the worst case, the second sum is equal to the first one and therefore the Fisher information becomes zero; in the best case the absolute value is identical, but the sign reversed, leading to twice the value. In this case the accuracy for the complex signal model is reached, even though only the real part of the signal was available (see below). On average across all possible signal phases, the second sum is zero; the first sum is the average Fisher information in this sense. A closed form expression can be given for this part, using $t_n = (n + n_0)T$, with $n_0 = -\frac{N-1}{2}$ [24].

$$I_{11} = \frac{A^2 T^2}{2\sigma^2} \frac{N(N^2-1)}{12} \quad (19)$$

This can be used as a rough estimate for the CRB:

$$\text{Var}(\omega) \geq I^{-1} \approx \frac{\sigma^2}{A^2 T^2} \frac{24}{N(N^2-1)} \quad (20)$$

The visualization of the exact result in Fig. 7 is centered on this rough estimate, and the result shows that it is a reasonable approximation. For a complex valued sinusoid with no offset the derivation leads to an exact closed form solution [5]. The approximation above is equal to twice the value obtained for the complex valued case, which is not surprising as only half the number of independent noisy measurements is used.

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