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A general extending and constraining procedure for linear iterative methods

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Algebraic reconstruction techniques (ARTs), on both their successive and simultaneous formulations, have been developed since the early 1970s as efficient 'row-action methods' for solving the image-reconstruction problem in computerized tomography. In this respect, two important development directions were concerned with, first, their extension to the inconsistent case of the reconstruction problem and, second, their combination with constraining strategies, imposed by the particularities of the reconstructed image. In the first part of this paper, we introduce extending and constraining procedures for a general iterative method of an ART type and we propose a set of sufficient assumptions that ensure the convergence of the corresponding algorithms. As an application of this approach, we prove that Cimmino's simultaneous reflection method satisfies this set of assumptions, and we derive extended and constrained versions for it. Numerical experiments with all these versions are presented on a head phantom widely used in the image reconstruction literature. We also consider hard thresholding constraining used in sparse approximation problems and apply it successfully to a 3D particle image-reconstruction problem.

Keywords: algebraic reconstruction techniques; inconsistent least-squares problems; constraining strategies; Cimmino algorithm; Cimmino extended algorithm; hard thresholding operator

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1. Introduction

Many classes of 'real-world problems' give rise, after appropriate discretizations, to big, sparse and ill-conditioned linear systems of equations of the form $\mathbf{A}x = b$, where the $m \times n$ matrix $\mathbf{A}$ contains information concerning the problem, whereas $b \in \mathbb{R}^m$ represents measured 'effects' produced by the unknown 'cause' $x \in \mathbb{R}^n$. But, due to inevitable measurement errors, the 'effect' b may go out of the 'range of action' of the problem information matrix $\mathbf{A}$, such that the above system of equations becomes inconsistent and must be reformulated in the least-squares sense:

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find $x \in \mathbb{R}^n$ such that

$$\| \mathbf{A}x - b \| = \min\{\| \mathbf{A}z - b \|, z \in \mathbb{R}^n\}, \tag{1}$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^n$.

Remark 1 Concerning the matrix involved in Equation (1), in the remainder of this paper, we suppose that it has non-zero rows $\mathbf{A}_i$ and columns $\mathbf{A}^j$, that is,

$$\mathbf{A}_i \neq 0, \quad i = 1, \ldots, m, \quad \mathbf{A}^j \neq 0, \quad j = 1, \ldots, n. \tag{2}$$

These assumptions are not essential restrictions of the generality of problem (1) because if $\mathbf{A}$ has null rows and/or columns, it can be easily proved that they can be eliminated without affecting its set of classical solutions (denoted by $S(\mathbf{A}; b)$ in the consistent case) or least-squares solutions (denoted by $\mathrm{LSS}(\mathbf{A}; b)$ in the inconsistent case).

A very important example of such problems (tackled by the numerical experiments in the last section of this paper) is the image reconstruction from projections in computerized tomography. Its algebraic mathematical model, although essentially based on an integral equation formulation, gives rise after the 'rays × pixels' discretization procedure (for details, see [6,15]) to least-squares problems of the form (1). For the numerical solution of these problems, a class of algebraic reconstruction techniques (ARTs) were developed in the last 40 years (see [6] and references therein). These methods are iterative 'row-action' algorithms (i.e. they use rows or blocks of rows of the system matrix A in each iteration, without changing the values of its entries or its structure; see [6]) and are 'classified' according to the way in which the rows/blocks of rows are 'visited' in each iteration:

(i) successive ART, having as the standard method the Kaczmarz algorithm [18] and
(ii) simultaneous ART, having as the standard method Cimmino's algorithm [9].

According to these standard algorithms, in this paper, we consider ART-like methods of the following general form.

ALGORITHM 1 General ART (GenART)
Initialization: $x^0 \in \mathbb{R}^n$.
Iterative step:

$$x^{k+1} = \mathbf{T}x^k + \mathbf{R}b, \tag{3}$$

where $\mathbf{T}$ *and* $\mathbf{R}$ *are* $n \times n$, *respectively*, $n \times m$, *real matrices.*

Remark 2 We introduce the following assumption on the matrices $\mathbf{T}$ and $\mathbf{R}$: they have an explicit expression in terms of the rows of $\mathbf{A}$ and components present on the right-hand side of Equation (1), that is, if we change $\mathbf{A}$ and b to $\bar{\mathbf{A}}$ and $\bar{b}$, in a way similar to that used in Equation (3), we can define an iterative process of the form $x^{k+1} = \bar{\mathbf{T}}x^k + \bar{\mathbf{R}}\bar{b}$. Examples in this sense are given by the projection algorithms appearing in image reconstruction from projections: Kaczmarz, Cimmino, Landweber, Diagonal Weighting (DW), Simultaneous Algebraic Reconstruction Technique (SART), etc. (see e.g. [5–7,12,15,17,19,20,30] and references therein).

Almost all these projection algorithms generate sequences convergent to a solution of problem (1) in the consistent case, whereas in the inconsistent one, the sequence $(x^k)_{k\geq 0}$ still converges, but the limit is not an element of $\mathrm{LSS}(\mathbf{A}; b)$ any more. In this respect , their extensions have been designed for the inconsistent case of Equation (1), which are based on relaxation parameters, column relaxations or supplementary steps introduced in the iteration (see [3,10,21,23,25,29] and references therein). Moreover, for problems related to image reconstruction in computerized tomography, their specific iteration step like Equation (3) was combined with a constraining strategy, usually acting on the components of the successive approximations $x^k = (x_1^k, \dots, x_n^k)^{\mathrm{T}}$ [19,23,26].

In this paper, we analyse from these points of view the general ART algorithm (3). The paper is organized as follows. In Section 2, we present the essential assumptions on the matrices **T** and **R** from Equation (3), which ensure the possibility to both extend it to inconsistent problems and combine it with a class of constraining strategies. Moreover, we prove that these assumptions are sufficient for obtaining convergence results for the extended and constrained versions of the general ART method. As an application of these main results of the paper, in Section 3, we prove that Cimmino's reflection algorithm [9] satisfies all the above assumptions and we derive its extended and constrained versions. Even though Cimmino's method was created many years ago, it can now be regarded as a special case of the Landweber method, such that the constrained Cimmino algorithm can be retrieved as a particular 'projected Landweber method' [1] or as a 'gradient projection' algorithm [14]. The fact that it was used in this section as an application for the considerations given in Section 2 is only a historical point of view.

The extension procedure proposed in Equations (36)–(38) differs from the older 'multi-step' methods [3,29] or methods that use the associated augmented system (which is always consistent) in the inconsistent case for Equation (1). As motivation, we consider the following two aspects: first, the modification on the right-hand side in Equation (37) is included in the iteration of the extended algorithm and thus in the global convergence of the algorithm (so accumulation of errors does not appear due to approximate solutions in the different steps of the 'multi-step' methods) and, second, the fact that by acting on the initial problem, the extended method (36)–(38) is influenced by its condition number and not by the squared one, as in the case of an augmented system or a normal equation (see [2] and the numerical experiments in [21]). Moreover, we would like to point out that the extending and constraining approach developed under assumptions (36)–(38) is quite general and can be applied to other algorithms too (e.g. DW or SART algorithm [17]), for which it will be possible to prove these assumptions. Moreover, once these assumptions are verified, we get three new algorithms: extended, constrained and constrained extended.

The last section of the paper is devoted to experiments with all these versions on a phantom widely used in the literature. Moreover, we consider different constraining strategies, including hard thresholding, and compare these in the context of particle image reconstruction.

2. The general extending and constraining procedures

First, we introduce some notation. The spectrum and spectral radius of a square matrix are denoted by $\sigma(\mathbf{B})$ and $\rho(\mathbf{B})$, respectively. By $\mathbf{A}^{\mathrm{T}}$, $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$, we denote the transpose, null space and range of $\mathbf{A}$. $P_S(x)$ is the orthogonal (Euclidean) projection onto a vector subspace S of $\mathbb{R}^n$. $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$, and $S(\mathbf{A}; b)$ and $\mathrm{LSS}(\mathbf{A}; b)$ stand for the set of classical or least-squares solutions of Equation (1), respectively. By x_{LS}, we denote the (unique) solution with minimal Euclidean norm (in both cases). In the general case for Equation (1), the following properties are known:

$$b = P_{\mathcal{R}(\mathbf{A})}(b) + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b), \tag{4}$$

$$\mathrm{LSS}(\mathbf{A}; b) = \{P_{\mathcal{N}(\mathbf{A})}(z) + x_{\mathrm{LS}}, z \in \mathbb{R}^n\}, \quad x \in \mathrm{LSS}(\mathbf{A}; b) \Leftrightarrow \mathbf{A}x = P_{\mathcal{R}(\mathbf{A})}(b), \tag{5}$$

$$S(\mathbf{A}; b) = \{P_{\mathcal{N}(\mathbf{A})}(z) + x_{\mathrm{LS}}, z \in \mathbb{R}^n\} \quad \text{and} \quad x \in S(\mathbf{A}; b) \Leftrightarrow \mathbf{A}x = b. \tag{6}$$

Moreover, x_{LS} is the unique element of $\text{LSS}(\mathbf{A}; b)$ (or $S(\mathbf{A}; b)$) which belongs to the subspace $\mathcal{R}(\mathbf{A}^{\text{T}})$. The spectral norm of $\mathbf{A}$ is defined by

$$\| \mathbf{A} \| = \sup_{x \neq 0} \frac{\| \mathbf{A}x \|}{\| x \|} = \sup_{\|x\|=1} \| \mathbf{A}x \|. \tag{7}$$

Now, we introduce the following *basic* assumptions on the above-considered matrices $\mathbf{T}$ and $\mathbf{R}$:

$$\mathbf{I} - T = RA, \tag{8}$$

$$\text{if } x \in \mathcal{N}(\mathbf{A}), \text{ then } \mathbf{T}x = x \in \mathcal{N}(\mathbf{A}), \tag{9}$$

$$\text{if } x \in \mathcal{R}(\mathbf{A}^{\text{T}}), \text{ then } \mathbf{T}x \in \mathcal{R}(\mathbf{A}^{\text{T}}), \tag{10}$$

$$\forall y \in \mathbb{R}^m, \quad \mathbf{R}y \in \mathcal{R}(\mathbf{A}^{\text{T}}), \tag{11}$$

$$\text{if } \widetilde{\mathbf{T}} = \mathbf{T}P_{\mathcal{R}(\mathbf{A}^{\text{T}})}, \text{ then } \| \widetilde{\mathbf{T}} \| < 1. \tag{12}$$

Proposition 2.1 *If Equations (8)–(12) hold, then the following are true:*

(i) $\mathbf{I} - \tilde{\mathbf{T}}$ *is invertible and the* $n \times m$ *matrix* $\mathbf{G}$ *defined by*

$$\mathbf{G} = (\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R} \tag{13}$$

satisfies

$$\mathbf{AGA} = \mathbf{A} \quad \text{and} \quad \mathbf{G}P_{\mathcal{R}(\mathbf{A})}(b) = x_{\text{LS}}. \tag{14}$$

(ii) *The matrix* $\mathbf{T}$ *has the properties*

$$\| \mathbf{T}x \| = \| x \| \quad \text{if and only if } x \in \mathcal{N}(\mathbf{A}) \tag{15}$$

and

$$\| \mathbf{T} \| \leq 1. \tag{16}$$

(iii) *For the approximations* $x^k, k \geq 0$, *generated with algorithm (3), we have*

$$P_{\mathcal{N}(\mathbf{A})}(x^k) = P_{\mathcal{N}(\mathbf{A})}(x^0), \quad \forall k \geq 0. \tag{17}$$

Proof (i) From Equation (12) [2], it results that the matrix $\mathbf{I} - \widetilde{\mathbf{T}}$ is invertible and

$$(\mathbf{I} - \widetilde{\mathbf{T}})^{-1} = \sum_{i \geq 0} \widetilde{\mathbf{T}}^i. \tag{18}$$

From the definition of $\widetilde{\mathbf{T}}$ in Equations (12) and (9) [30], we get

$$\mathbf{T} = P_{\mathcal{N}(\mathbf{A})} + \widetilde{\mathbf{T}}, \widetilde{\mathbf{T}}P_{\mathcal{N}(\mathbf{A})} = P_{\mathcal{N}(\mathbf{A})}\widetilde{\mathbf{T}} = 0. \tag{19}$$

Then, from Equations (8), (13), (19) and (18), we successively obtain

$$\begin{aligned}\mathbf{AGA} &= \mathbf{A}(\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}A = \mathbf{A}(\mathbf{I} - \widetilde{\mathbf{T}})^{-1}(\mathbf{I} - \mathbf{T}) \\ &= \mathbf{A}(\mathbf{I} - \widetilde{\mathbf{T}})^{-1}((\mathbf{I} - \widetilde{\mathbf{T}}) - P_{\mathcal{N}(\mathbf{A})}) = \mathbf{A} - \mathbf{A}(\mathbf{I} - \widetilde{\mathbf{T}})^{-1}P_{\mathcal{N}(\mathbf{A})} = \mathbf{A},\end{aligned}$$

that is, the first equality in Equation (14). Then, because $P_{\mathcal{R}(\mathbf{A})}(b) \in \mathcal{R}(\mathbf{A})$, we get from the first equality in Equation (14) $\mathbf{AG}P_{\mathcal{R}(\mathbf{A})}(b) = P_{\mathcal{R}(\mathbf{A})}(b)$, which means that $x^* = \mathbf{G}P_{\mathcal{R}(\mathbf{A})}(b) \in$

LSS$(\mathbf{A}; b)$ [2]. But from Equations (11) and (18) and the definition of $\widetilde{\mathbf{T}}$, it results that $x^* \in \mathcal{R}(\mathbf{A}^{\mathrm{T}})$, that is, $x^* = x_{\mathrm{LS}}$, thus by the unicity of x_{LS}, which proves the second equality in Equation (14).

(ii) The 'if' part results directly from Equation (9). For the 'only if' one, let $x \in \mathbb{R}^n$ be such that $\| \mathbf{T}x \| = \| x \|$ holds. Then, if $x = x' + x'' = P_{\mathcal{N}(\mathbf{A})}(x) + P_{\mathcal{R}(\mathbf{A}^{\mathrm{T}})}(x)$, and $x'' \neq 0$, from Equations (9) and (10), we get $\mathbf{T}x' \in \mathcal{N}(\mathbf{A})$ and $\mathbf{T}x'' \in \mathcal{R}(\mathbf{A}^{\mathrm{T}})$. Thus, by also using Equation (12), we successively obtain

$$\begin{aligned}\| \mathbf{T}x \|^2 = \| \mathbf{T}x' \|^2 + \| \mathbf{T}x'' \|^2 &\leq \| x' \|^2 + \| \widetilde{\mathbf{T}} \|^2 \quad \| x'' \|^2 \\ &< \| x' \|^2 + \| x'' \|^2 = \| x \|^2, \end{aligned} \tag{20}$$

which contradicts our initial assumption about the vector x. It follows that $x'' = 0$, that is, $x \in \mathcal{N}(\mathbf{A})$. The inequality (16) results from Equation (20) for an arbitrary $x \in \mathbb{R}$ (in which case the last inequality is not any more strict).

(iii) We use the mathematical induction. Let us suppose that $k \geq 0$ is such that Equation (17) holds. For $k+1$, we have, by also using Equations (3) and (19)

$$x^{k+1} = \mathbf{T}x^k + \mathbf{R}b = P_{\mathcal{N}(\mathbf{A})}(x^k) + \widetilde{\mathbf{T}}x^k + \mathbf{R}b.$$

But from Equations (10) and (11), we obtain that $\widetilde{\mathbf{T}}x^k + \mathbf{R}b \in \mathcal{R}(\mathbf{A}^{\mathrm{T}})$, that is, $P_{\mathcal{N}(\mathbf{A})}(x^{k+1}) = P_{\mathcal{N}(\mathbf{A})}(x^k) = P_{\mathcal{N}(\mathbf{A})}(x^0)$, which completes the proof. ■

The convergence properties of the algorithm GenART (3) are given in the following result.

THEOREM 2.2 *Let us suppose that the matrices* $\mathbf{T}$ *and* $\mathbf{R}$ *satisfy assumptions* (8)–(12). *Then, for any* $x^0 \in \mathbb{R}^n$, *the sequence* $(x^k)_{k\geq 0}$ *generated with algorithm* (3) *converges and*

$$\lim_{k\to\infty} x^k = P_{\mathcal{N}(\mathbf{A})}(x^0) + \mathbf{G}b. \tag{21}$$

If problem (1) *is consistent, then*

$$\mathbf{G}b = x_{\mathrm{LS}} \quad \text{and} \quad \lim_{k\to\infty} x^k = P_{\mathcal{N}(\mathbf{A})}(x^0) + x_{\mathrm{LS}} \in S(\mathbf{A}; b). \tag{22}$$

Proof Let $e^k = x^k - (P_{\mathcal{N}(\mathbf{A})}(x^0) + \mathbf{G}b)$ be the error vector at iteration k (see Equation (21)). Using Equations (3), (13), (17) and (20), we successively obtain

$$\begin{aligned} e^k &= x^k - (P_{\mathcal{N}(\mathbf{A})}(x^0) + \mathbf{G}b) = x^k - [P_{\mathcal{N}(\mathbf{A})}(x^0) + [(\mathbf{I} - \widetilde{\mathbf{T}}) + \widetilde{\mathbf{T}}](\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}b] \\ &= \widetilde{\mathbf{T}}x^{k-1} - \widetilde{\mathbf{T}}(\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}b = \widetilde{\mathbf{T}}(x^{k-1} - P_{\mathcal{N}(\mathbf{A})}(x^0) - \mathbf{G}b) = \widetilde{\mathbf{T}}e^{k-1}, \end{aligned}$$

and by a recursive argument,

$$e^k = \widetilde{\mathbf{T}}^k e^0, \quad \forall k \geq 0. \tag{23}$$

But according to Equation (12), we get that $\lim_{k\to\infty} e^k = 0$, from which we get Equation (21). The second part of the theorem (22) results from Proposition 2.1(i). ■

THEOREM 2.3 *Let* x^* *be the limit point in Equation* (21). *Then, we have the a priori estimate*

$$\|x^k - x^*\| \leq \frac{\kappa^k}{1-\kappa}\|x^0 - x^1\| \tag{24}$$

and the a posteriori estimate

$$\|x^{k+1} - x^*\| \leq \frac{\kappa}{1-\kappa}\|x^{k+1} - x^k\|, \tag{25}$$

where $\kappa = \|\widetilde{\mathbf{T}}\|$. *In particular, the convergence rate of sequence* $(x^k)_{k\geq 0}$ *is linear.*

Proof Let $(x^k)_{k\geq 0}$ be the sequence generated by GenART for an arbitrary initial approximation $x^0 \in \mathbb{R}^n$ and suppose that the matrices **T** and **R** satisfy assumptions (8)–(12). Then, using Equation (17), we can rewrite Equation (3) as

$$x^{k+1} = \widetilde{\mathbf{T}}x^k + P_{\mathcal{N}(\mathbf{A})}(x^0) + \mathbf{R}b =: F(x^k), \tag{26}$$

since we can decompose **T** according to Equation (19). The mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a contraction with the Lipschitz constant $\kappa := \|\widetilde{\mathbf{T}}\|$. Banach's fixed-point theorem asserts, additionally to the convergence of sequence $(x^k)_{k\geq 0}$ to a fixed point of F, the estimates in Equations (24) and (25). ■

Remark 3 We claim that the above set of sufficient assumptions (8)–(12) is also necessary to obtain the results in Proposition 2.1 and Theorem 2.2, but we do not have a rigorous proof of this statement yet.

According to Remark 2 given in Section 1, let **U** and **S** be the $m \times m$, respectively, $m \times n$, matrices, similar to **T** and **R** from Equation (3), respectively, but for the (always consistent) system

$$\mathbf{A}^{\mathrm{T}}y = 0. \tag{27}$$

Example 2.4 For the classical Kaczmarz successive projection method, the $n \times n$ matrix **T** is given by

$$\mathbf{T}(x) = (f_1 \circ \cdots \circ f_m)(x), \quad f_i(x) = P_i(x) + \frac{b_i}{\| \mathbf{A}_i \|^2}\mathbf{A}_i,$$
$$P_i(x) = x - \frac{\langle x, \mathbf{A}_i \rangle}{\| \mathbf{A}_i \|^2}\mathbf{A}_i, \qquad x \in \mathbb{R}^n,\ i = 1, \dots, m, \tag{28}$$

and **R** is the $n \times m$ matrix of which the ith column is $(1/\| \mathbf{A}_i \|^2)P_1 \cdots P_{i-1}$ (where P_0 is, by definition, the identity). In this case, the above matrices **U** and **S** are obtained by just applying the above construction for system (27), that is, by replacing m with n and the rows of **A** with its columns and setting $b = 0$, namely

$$\mathbf{U}(y) = (\phi_1 \circ \cdots \circ \phi_n)(y), \quad \phi_i(y) = y - \frac{\langle x, \mathbf{A}^j \rangle}{\| \mathbf{A}^j \|^2}\mathbf{A}^j, \quad y \in \mathbb{R}^m,\ j = 1, \dots, n, \tag{29}$$

and **S** is the $m \times n$ matrix of which the jth column is $(1/\| \mathbf{A}^j \|^2)\phi_1 \cdots \phi_{j-1}$ (where ϕ_0 is, by definition, the identity).

Then, the corresponding algorithm of the form (3) with **U** and **S** will be written as

$$y^{k+1} = \mathbf{U}y^k + \mathbf{S} \cdot 0 = \mathbf{U}y^k, \quad \forall k \geq 0, \tag{30}$$

with $y^0 \in \mathbb{R}^m$ being the initial approximation. Our general assumptions (8)–(12) and Proposition 2.1 will assign the following properties to the matrix **U**:

$$\mathbf{U}(\mathcal{N}(\mathbf{A}^{\mathrm{T}})) \subset \mathcal{N}(\mathbf{A}^{\mathrm{T}}), \quad \mathbf{U}(\mathcal{R}(\mathbf{A})) \subset \mathcal{R}(\mathbf{A}), \tag{31}$$

$$\text{if } \widetilde{\mathbf{U}} = \mathbf{U}P_{\mathcal{R}(\mathbf{A})},\ \text{then } \mathbf{U} = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})} \oplus \widetilde{\mathbf{U}} \quad \text{and} \quad P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})} = 0, \tag{32}$$

$$\mathbf{U}^k = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})} \oplus \widetilde{\mathbf{U}}^k, \quad \| \widetilde{\mathbf{U}} \| < 1 \quad \text{and} \quad P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^k) = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0), \quad \forall k \geq 0. \tag{33}$$

Moreover, according to Theorem 2.2, the following convergence result will hold for algorithm (30).

THEOREM 2.5 *For any $y^0 \in \mathbb{R}^m$, the sequence $(y^k)_{k\geq 0}$ generated with algorithm (30) converges and*

$$\lim_{k\to\infty} y^k = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0). \tag{34}$$

Proof Let

$$\varepsilon^k = y^k - P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0) \tag{35}$$

be the error vector at iteration k (see Equation (34)). Using Equations (30), (32), (33) and (35), we successively obtain

$$\begin{aligned}\varepsilon^k &= \mathbf{U}y^{k-1} - P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0) = \widetilde{\mathbf{U}}y^{k-1} + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^{k-1}) - P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0) = \widetilde{\mathbf{U}}y^{k-1}\\ &= \widetilde{\mathbf{U}}(y^{k-1} - P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(y^0)) = \widetilde{\mathbf{U}}e^{k-1}.\end{aligned}$$

But from Equation (33), we get $\lim_{k\to\infty} \varepsilon^k = 0$, from which Equation (34) holds and completes the proof. ∎

If $y^0 = b$, from Equation (34), we would get $\lim_{k\to\infty} y^k = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b)$, thus $\lim_{k\to\infty}(b - y^k) = P_{\mathcal{R}(\mathbf{A})}(b)$. This simple observation allows us to consider the following extension of the general algorithm GenART.

ALGORITHM 2 Extended General ART (EGenART)
Initialization: $x^0 \in \mathbb{R}^n, y^0 = b$.
Iterative step:

$$y^{k+1} = \mathbf{U}y^k, \tag{36}$$

$$b^{k+1} = b - y^{k+1}, \tag{37}$$

$$x^{k+1} = \mathbf{T}x^k + \mathbf{R}b^{k+1}. \tag{38}$$

THEOREM 2.6 *Let us suppose that the matrices* $\mathbf{T}$ *and* $\mathbf{R}$ *satisfy Equations (8)–(12) and* $\mathbf{U}$ *is as described earlier. Then, for any $x^0 \in \mathbb{R}^n$, the sequence $(x^k)_{k\geq 0}$ generated with algorithms (36)–(38) converges and*

$$\lim_{k\to\infty} x^k = P_{\mathcal{N}(\mathbf{A})}(x^0) + x_{\mathrm{LS}} \in \mathrm{LSS}(\mathbf{A}; b). \tag{39}$$

Proof Let $e^k = x^k - (P_{\mathcal{N}(\mathbf{A})}(x^0) + x_{\mathrm{LS}})$ be the error vector at iteration $k \geq 1$ (see Equation (39)). Using Equations (13), (14), (17), (19), (37) and (38), we successively obtain

$$\begin{aligned}e^k &= x^k - (P_{\mathcal{N}(\mathbf{A})}(x^0) + x_{\mathrm{LS}})\\ &= \mathbf{T}x^{k-1} + \mathbf{R}b^k - [P_{\mathcal{N}(\mathbf{A})}(x^0) + [(\mathbf{I} - \widetilde{\mathbf{T}}) + \widetilde{\mathbf{T}}](\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}P_{\mathcal{R}(\mathbf{A})}(b)]\\ &= P_{\mathcal{N}(\mathbf{A})}(x^{k-1}) + \widetilde{\mathbf{T}}x^{k-1} + \mathbf{R}b^k - P_{\mathcal{N}(\mathbf{A})}(x^0) - \mathbf{R}P_{\mathcal{R}(\mathbf{A})}(b) - \widetilde{\mathbf{T}}\mathbf{G}P_{\mathcal{R}(\mathbf{A})}(b)\\ &= \widetilde{\mathbf{T}}x^{k-1} + \mathbf{R}b^k - \mathbf{R}P_{\mathcal{R}(\mathbf{A})}(b) - \widetilde{\mathbf{T}}\mathbf{G}P_{\mathcal{R}(\mathbf{A})}(b)\\ &= \widetilde{\mathbf{T}}(x^{k-1} - P_{\mathcal{N}(\mathbf{A})}(x^0) - x_{\mathrm{LS}}) + \mathbf{R}(b - y^k - P_{\mathcal{R}(\mathbf{A})}(b)).\end{aligned} \tag{40}$$

Because $y^0 = b$, from Equations (4) and (30), we get

$$y^k = \mathbf{U}y^{k-1} = \widetilde{\mathbf{U}}y^{k-1} + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b), \quad \forall k \geq 1. \tag{41}$$

From Equations (4) and (41), we obtain

$$\mathbf{R}(b - y^k - P_{\mathcal{R}(\mathbf{A})}(b)) = -\mathbf{R}\widetilde{\mathbf{U}}y^{k-1},$$

which gives us, according to Equation (40),

$$e^k = \widetilde{\mathbf{T}}e^{k-1} - \mathbf{R}\widetilde{\mathbf{U}}y^{k-1}, \quad \forall k \geq 1. \tag{42}$$

A recursive argument gives us from Equation (42)

$$e^k = \widetilde{\mathbf{T}}^k e^0 - \sum_{i=1}^{k} \widetilde{\mathbf{T}}^{i-1}\mathbf{R}\widetilde{\mathbf{U}}y^{k-i}, \quad \forall k \geq 1. \tag{43}$$

Now, from Equations (32) and (36), we get $\widetilde{\mathbf{U}}y^0 = \widetilde{\mathbf{U}}b$, $\widetilde{\mathbf{U}}y^1 = \widetilde{\mathbf{U}}\mathbf{U}y^0 = \widetilde{\mathbf{U}}(\widetilde{\mathbf{U}} + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})})y^0 = \widetilde{\mathbf{U}}^2 b$ and, in general, by mathematical induction,

$$\widetilde{\mathbf{U}}y^j = \widetilde{\mathbf{U}}^{j+1}b, \quad \forall j \geq 0. \tag{44}$$

Now, using Equation (44) and taking Euclidean and spectral norms in Equation (43), we obtain

$$\begin{aligned} \| e^k \| &\leq \| \widetilde{\mathbf{T}} \|^k e^0 + \left(\sum_{i=1}^{k} \| \widetilde{\mathbf{T}} \|^{i-1} \| \widetilde{\mathbf{U}} \|^{k-i+1} \right) \| b \| \| \mathbf{R} \| \\ &\leq \delta^k \| e^0 \| + \left(\sum_{i=1}^{k} \delta^{i-1}\delta^{k-(i-1)} \right) \| b \| \| \mathbf{R} \| = \delta^k \| e^0 \| + k\delta^k \| b \| \| \mathbf{R} \|, \quad \forall k \geq 0, \end{aligned} \tag{45}$$

where δ is defined by (see Equations (12) and (33))

$$\delta = \max\{\| \widetilde{\mathbf{T}} \|, \| \widetilde{\mathbf{U}} \|\} \in [0, 1). \tag{46}$$

According to Equations (45) and (46), we obtain $\lim_{k\to\infty} e^k = 0$, from which Equation (39) holds and completes the proof. ■

Remark 4 A different extension procedure has been proposed in [11]. It uses similar ideas in the convergence proof, but under different initial assumptions.

Remark 5 By using Equations (4), (13), (39) and the second equality in Equation (14), we obtain that the limit (21) of the sequence $(x^k)_{k\geq 0}$ generated by Equation (3) can be written as

$$\begin{aligned} \lim_{k\to\infty} x^k &= P_{\mathcal{N}(\mathbf{A})}(x^0) + (\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}(P_{\mathcal{R}(\mathbf{A})}(b) + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b)) \\ &= P_{\mathcal{N}(\mathbf{A})}(x^0) + x_{\mathrm{LS}} + \Delta, \quad \text{with } \Delta = (\mathbf{I} - \widetilde{\mathbf{T}})^{-1}\mathbf{R}P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b) = \mathbf{G}P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b). \end{aligned} \tag{47}$$

Koltracht and Lancaster [19] considered a constraining function, $C : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, with a closed image $\mathrm{Im}(C) \subset \mathbb{R}^n$ and the properties

$$\| Cx - Cy \| \leq \| x - y \|, \tag{48}$$

$$\text{if } \| Cx - Cy \| = \| x - y \|, \quad \text{then } Cx - Cy = x - y, \tag{49}$$

$$\text{if } y \in \mathrm{Im}(C), \quad \text{then } y = Cy. \tag{50}$$

Example 2.7 'Box-constraining' function

$$(Cx)_i = \begin{cases} x_i, & x_i \in [\alpha_i, \beta_i], \\ \alpha_i, & x_i < \alpha_i, \\ \beta_i, & x_i > \beta_i, \end{cases} \tag{51}$$

that is, C is the orthogonal projection onto the closed convex set $V = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset \mathbb{R}^n$, for which it is well known that Equations (48)–(50) hold [16] and its image is closed ($\mathrm{Im}(C) = V$).

Example 2.8 A more general constraining function is the *hard thresholding operator* reported in [4], which has the following general form ($\alpha \geq 0$):

$$H_\alpha(y) = (h_\alpha(y_1), \ldots, h_\alpha(y_n)), \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n \tag{52}$$

with

$$h_\alpha(y_i) = \begin{cases} 0, & |y_i| < \alpha \\ y_i, & y_i \in (-\infty, -\alpha] \cup [\alpha, \infty) \end{cases}, \quad i = 1, \ldots, n. \tag{53}$$

From the definitions (52) and (53), it results that $\mathrm{Im}(H_\alpha) \subset \mathbb{R}^n$ is closed and $h_\alpha(h_\alpha(x_i)) = h_\alpha(x_i)$, $\forall x_i \in \mathbb{R}$, $\forall i = 1, \ldots, n$. Thus, $H_\alpha(H_\alpha(x)) = H_\alpha(x)$, $\forall x \in \mathbb{R}^n$, that is, assumption (50). Unfortunately, the function H_α does not satisfy Equations (48) and (49).

ALGORITHM 3 Constrained General ART (CGenART)
Initialization: $x^0 \in \mathbb{R}^n$.
Iterative step:

$$x^{k+1} = C[\mathbf{T}x^k + \mathbf{R}b]. \tag{54}$$

THEOREM 2.9 *Let us suppose that the matrices* **T** *and* **R** *satisfy Equations* (*8*)–(*12*), *the constraining function C satisfies Equations* (*48*)–(*50*) *and the set* $\mathcal{V}^*$, *defined by*

$$\mathcal{V}^* = \{y \in \mathrm{Im}(C),\ y - \Delta \in \mathrm{LSS}(A; b)\}, \tag{55}$$

is non-empty. Then, for any $x^0 \in \mathrm{Im}(C)$, *the sequence* $(x^k)_{k \geq 0}$ *generated by the algorithm CGenART converges and its limit belongs to the set* $\mathcal{V}^*$.

Proof We follow the ideas of the proof given in [19], but for the general matrices **T** and **R** from Equation (3), satisfying assumptions (8)–(12). For this, first, we show that if $h \in \mathrm{Im}(C)$ and

$$g = C[\mathbf{T}h + \mathbf{R}b], \tag{56}$$

then for any $y \in \mathcal{V}^*$,

$$\| g - y \| \leq \| h - y \| \tag{57}$$

and either

$$\| g - y \| < \| h - y \| \tag{58}$$

or

$$g = h \in \mathcal{V}^*. \tag{59}$$

In this respect, for an arbitrary fixed $y \in \mathcal{V}^*$, according to Equations (5) and (55), let $z = P_{\mathcal{N}(\mathbf{A})}(z) + x_{\mathrm{LS}} = y - \Delta \in \mathrm{LSS}(\mathbf{A}; b)$, and $\xi = x_{\mathrm{LS}} + \Delta$ (i.e. the limit from Equation (47) for

$x^0 = 0$). Because $z - x_{\mathrm{LS}} \in \mathcal{N}(\mathbf{A})$, from Equation (9), we obtain $\mathbf{T}(z - x_{\mathrm{LS}}) = z - x_{\mathrm{LS}}$. Thus, by also using Equations (14), (18) and (44), we get $\xi = x_{\mathrm{LS}} + \Delta = \mathbf{G}(P_{\mathcal{R}(\mathbf{A})}(b) + P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b)) = \mathbf{G}b$. Thus, by also using Equations (11), (13) and (19), we successively obtain

$$\begin{aligned}(\mathbf{I}-\mathbf{T})y &= (\mathbf{I}-\mathbf{T})(z+\Delta) = (\mathbf{I}-\mathbf{T})(z+\xi-x_{\mathrm{LS}}) = (\mathbf{I}-\mathbf{T})\xi\\ &= (\mathbf{I}-\mathbf{T})\mathbf{G}b = [+(\mathbf{I}-\widetilde{\mathbf{T}}) - P_{\mathcal{N}(\mathbf{A})}](\mathbf{I}-\widetilde{\mathbf{T}})^{-1}\mathbf{R}b\\ &= \mathbf{R}b - P_{\mathcal{N}(\mathbf{A})}(\mathbf{I}-\widetilde{\mathbf{T}})^{-1}\mathbf{R}b = \mathbf{R}b,\end{aligned} \tag{60}$$

where in the last equality, we use the fact that $P_{\mathcal{N}(\mathbf{A})}(\mathbf{I}-\widetilde{\mathbf{T}})^{-1}\mathbf{R}b = 0$, which holds from $(\mathbf{I}-\widetilde{\mathbf{T}})^{-1}\mathbf{R}b \in \mathcal{R}(\mathbf{A}^{\mathrm{T}})$ (see Equations (11), (12) and (18)). Now, Equations (56) and (60) give us

$$g - y = C[\mathbf{T}(h-y)+y] - Cy, \tag{61}$$

and thus from Equations (16), (48) and (50), we obtain

$$\| g - y \| = \| C[\mathbf{T}(h-y)+y] - Cy \| \le \| \mathbf{T}(h-y) \| \le \| h - y \|, \tag{62}$$

that is, Equation (57). If equality holds in Equation (57), from Equation (62), we obtain $\| \mathbf{T}(h-y) \| = \| h-y \|$, which according to Equations (9) and (15) gives us

$$h - y \in \mathcal{N}(\mathbf{A}), \text{ i.e. } \mathbf{T}(h-y) = h - y. \tag{63}$$

From Equations (49), (60), (61), (62) and (63), we conclude that $g = h$. Now, from the first relation in Equation (63), we get

$$h - \Delta = (y-\Delta) + (h-y) = P_{\mathcal{N}(\mathbf{A})}(y-\Delta) + x_{\mathrm{LS}} + (h-y) \in \mathcal{N}(\mathbf{A}) + x_{\mathrm{LS}} = \mathrm{LSS}(\mathbf{A}; b),$$

that is, $h \in \mathcal{V}^*$ (see Equation (55)). In order to prove the convergence of the sequence $(x^k)_{k\ge 0}$, we first observe that by applying Equations (56)–(57) with $g = x^{k+1}, h = x^k$, for any $y \in \mathcal{V}^*$, we obtain

$$\| x^{k+1} - y \| \le \| x^k - y \| \le \cdots \le \| x^0 - y \|, \quad \forall k \ge 0, \tag{64}$$

that is, the sequence $(x^k)_{k\ge 0}$ is bounded. Then, there exists a convergent subsequence $(x^{k_s})_{s\ge 0}$ such that

$$\lim_{s\to\infty} x^{k_s} = u \in \mathbb{R}^n. \tag{65}$$

We prove that

$$u \in \mathcal{V}^*. \tag{66}$$

If $v, z^s \in \mathbb{R}^n$ are defined by

$$v = C(Tu + Rb), \quad z^s = Tx^{k_s} + Rb, \tag{67}$$

then from Equations (65)–(67) and the continuity of C (see Equation (48)), we get

$$\lim_{s\to\infty} z^s = Tu + Rb, \quad \lim_{s\to\infty} C(z^s) = C(Tu+Rb) = v. \tag{68}$$

Because the set $\mathrm{Im}(C)$ is closed, $x^{k_s} \in \mathrm{Im}(C)$, $\forall s \ge 0$ (see Equations (54) and (67)), we obtain $u \in \mathrm{Im}(C)$, $v \in \mathrm{Im}(C)$, and according to Equations (58) and (59), only the following two cases are possible.

Case 1.

$$u = v \in \mathcal{V}^*, \tag{69}$$

which completes our proof for Equation (66).

Case 2.

$$\| v - y \| < \| u - y \|, \quad \forall y \in \mathcal{V}^*. \tag{70}$$

If Equation (70) holds and ϵ is such that $0 < \epsilon < \frac{1}{2}(\| u - y \| - \| v - y \|)$, we would get

$$\| v - y \| + \epsilon < \| u - y \| - \epsilon. \tag{71}$$

From Equations (65) and (68), we obtain

$$\lim_{s\to\infty} \| x^{k_s+1} - y \| = \| v - y \|, \quad \lim_{s\to\infty} \| x^{k_s} - y \| = \| u - y \|, \tag{72}$$

and thus for $\epsilon > 0$ defined earlier, there exist indices $s_1 \geq 1$ such that

$$| \| x^{k_{s_1}+1} - y \| - \| v - y \| | < \epsilon, \tag{73}$$

and $s_2 > s_1 + 1$ such that

$$| \| x^{k_{s_2}} - y \| - \| u - y \| | < \epsilon. \tag{74}$$

From the construction of s_1 and s_2, we get

$$k_{s_2} > k_{s_1+1} > k_{s_1} + 1, \tag{75}$$

thus from Equations (71), (73) and (74), we obtain

$$\| x^{k_{s_1+1}} - y \| < \| v - y \| + \epsilon < \| u - y \| - \epsilon < \| x^{k_{s_2}} - y \| . \tag{76}$$

But at the same time, Equations (64) and (75) gives us

$$\| x^{k_{s_2}} - y \| \leq \| x^{k_{s_1+1}} - y \|,$$

which contradicts Equation (76). It results that Equation (70) does not happen, thus only Equation (69) can occur.

Now, if $(x^{k'_s})_{s\geq 0}$ is an arbitrary convergent subsequence, that is,

$$\lim_{s\to\infty} x^{k'_s} = w, \tag{77}$$

from the previous considerations, we get $w \in \mathcal{V}^*$. We prove that $w = u$. Indeed, if this does not hold and ϵ_0 is such that

$$0 < \epsilon_0 < \| u - w \| - \epsilon_0, \tag{78}$$

we get (we denote by $B[x, r], B(x, r)$, the closed, respectively, open, ball in $\mathbb{R}^n$ with respect to the Euclidean norm)

$$B[u, \epsilon_0] \cap B[w, \epsilon_0] = \emptyset. \tag{79}$$

Let then $s \geq 1, x^{k_s}, x^{k'_s}$ have the properties

$$k_s > k'_s, \quad x^{k_s} \in B[u, \epsilon_0], \quad x^{k'_s} \in B[w, \epsilon_0]. \tag{80}$$

Then, from Equations (78)–(80), we obtain

$$\| x^{k_s} - w \| > \| u - w \| - \epsilon_0 > \epsilon_0 > \| x^{k'_s} - w \|,$$

which contradicts Equation (64) and completes the proof. ■

Remark 6 If problem (1) is consistent, then

$$\Delta = 0 \quad \text{and} \quad \mathcal{V}^* = S(\mathbf{A}; b) \cap \mathrm{Im}(C), \tag{81}$$

that is, the algorithm CGenART generates a 'constrained' solution of Equation (1).

Remark 7 We have to consider that all assumptions (48)–(50) are necessary in the proof of Theorem 2.9.

ALGORITHM 4 Constrained Extended General ART (CEGenART)
Initialization: $x^0 \in \mathrm{Im}(C), y^0 = b$.
Iterative step:

$$y^{k+1} = \mathbf{U}(y^k), \tag{82}$$

$$b^{k+1} = b - y^{k+1}, \tag{83}$$

$$x^{k+1} = C[\mathbf{T}x^k + \mathbf{R}b^{k+1}], \tag{84}$$

with $\mathbf{U}, \mathbf{T}$ *and* $\mathbf{R}$ *from Equations* (*3*), (*30*) *and C as in Equations* (*48*)–(*50*). *We suppose that at least one least-squares solution exists in* $\mathrm{Im}(C)$, *that is*, *the set* $\mathcal{V}$ *defined below is non-empty*:

$$\mathcal{V} = \mathrm{LSS}(\mathbf{A}; b) \cap \mathrm{Im}(C) \neq \emptyset. \tag{85}$$

For proving the convergence of the sequence $(x^k)_{k\geq 0}$ generated by the algorithm CEGenART, we follow the ideas of the proof reported in [26], but for the general matrices $\mathbf{T}$ and $\mathbf{R}$ from Equation (3), satisfying assumptions (8)–(12). In this respect, we first prove the following properties.

PROPOSITION 2.10 *Let us suppose that the matrices* $\mathbf{T}$ *and* $\mathbf{R}$ *satisfy Equations* (*8*)–(*12*), *the constraining function C satisfies Equations* (*48*) *and* (*50*) *and the set* $\mathcal{V}$ *satisfies Equation* (*85*). *Then*, *the following are true.*

(i) *The sequence* $(y^k)_{k\geq 0}$ *generated in step* (*82*) *of the algorithm CEGenArt satisfies*

$$y^{k+1} = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b) + \widetilde{\mathbf{U}}^{k+1} P_{\mathcal{R}(\mathbf{A})}(b). \tag{86}$$

(ii) *For any* $y \in \mathcal{V}$, *we have*

$$(\mathbf{I} - \mathbf{T})y = (\mathbf{I} - \mathbf{T})x_{\mathrm{LS}} = \mathbf{R}P_{\mathcal{R}(\mathbf{A})}(b), \tag{87}$$

and the sequence $(x^k)_{k\geq 0}$ *generated in step* (*84*) *of CEGenArt satisfies*

$$\| x^{k+1} - y \| \leq \| \mathbf{T}(x^k - y) - \mathbf{R}\widetilde{\mathbf{U}}^{k+1} P_{\mathcal{R}(\mathbf{A})}(b) \| . \tag{88}$$

(iii) *There exists a subsequence* $(x^{k_s})_{s\geq 0}$ *of* $(x^k)_{k\geq 0}$ *such that*

$$\lim_{s\to\infty} x^{k_s} = u \in \mathcal{V}. \tag{89}$$

Proof (i) From Equations (32), (33) and the first equation, we obtain the relations

$$y^{k+1} = P_{\mathcal{N}(\mathbf{A}^{\mathrm{T}})}(b) + \widetilde{\mathbf{U}}^{k+1}(b) \quad \text{and} \quad \widetilde{\mathbf{U}}^k(b) = \widetilde{\mathbf{U}}^k(P_{\mathcal{R}(\mathbf{A})}(b)), \tag{90}$$

from which Equation (86) directly holds.

(ii) Because $y \in \mathcal{V}$, we get, in particular, that $y \in \mathrm{LSS}(\mathbf{A}; b)$, and so we can write it as $y = P_{\mathcal{N}(\mathbf{A})}(y) + x_{\mathrm{LS}}$. Then, from Equations (9), (13) and (14), we get $(\mathbf{I} - \mathbf{T})y = (\mathbf{I} - \mathbf{T})x_{\mathrm{LS}} = (\mathbf{I} - \mathbf{T})\mathbf{G}P_{\mathcal{R}(\mathbf{A})}(b)$, from which Equation (87) holds with arguments similar to those for Equation (60).

For proving inequality (88), we first observe that for an arbitrary fixed $y \in \mathcal{V}$ from Equations (4), (50), (83), (84), (86) and (87), we obtain

$$\begin{aligned} x^{k+1} - y &= C[\mathbf{T}x^k + \mathbf{R}b^{k+1}] - y \\ &= C[\mathbf{T}x^k + (\mathbf{I} - \mathbf{T})y - \mathbf{R}\widetilde{\mathbf{U}}^{k+1}(P_{\mathcal{R}(\mathbf{A})}(b))] - Cy, \end{aligned} \tag{91}$$

from which Equation (88) holds by taking norms and using Equation (48).

(iii) From Equations (16), (88) and the second inequality in (33), we obtain

$$\| x^{k+1} - y \| \leq \| x^k - y \| + c\delta^{k+1}, \quad \forall k \geq 0, \tag{92}$$

with $c = \| \mathbf{R} \| \cdot \| P_{\mathcal{R}(\mathbf{A})}(b) \|$, $\delta = \| \widetilde{\mathbf{U}} \| \in [0, 1)$. By a recursive application of inequality (92), we obtain

$$\| x^{k+1} - y \| \leq \| x^0 - y \| + c\delta \frac{1 - \delta^{k+1}}{1 - \delta} < \| x^0 - y \| + \frac{c\delta}{1 - \delta}, \quad \forall k \geq 0,$$

which tells us that the sequence $(x^k)_{k \geq 0}$ is bounded. Thus, a subsequence $(x^{k_s})_{k_s \geq 0}$ of it exists such that $\lim_{s \to \infty} x^{k_s} = u$. Moreover, $u \in \mathrm{Im}(C)$ because $x^{k_s} \in \mathrm{Im}(C)$, $\forall s \geq 0$, and the set $\mathrm{Im}(C)$ is closed. For proving that $u \in \mathcal{V}$, it suffices to show that $u \in \mathrm{LSS}(\mathbf{A}; b)$ (see Equation (85)). For this, we follow exactly the proof of Lemma 4 reported in [26], by replacing the matrix Q with $\mathbf{T}$ from Equation (84). ■

THEOREM 2.11 *In the hypothesis of Proposition 2.10, the sequence $(x^k)_{k \geq 0}$ generated with the algorithm CEGenART (82)–(84) converges to an element of $\mathcal{V}$.*

Proof As in the proof of Lemma 5 reported in [26], we show that any convergent subsequence $(x^{\bar{k}_s})_{s \geq 0}$ of $(x^k)_{k \geq 0}$ has as limit the element u from Equation (89). ■

Remark 8 The assumption that the set $\mathcal{V}$ from Equation (85) is non-empty is directly connected to the (level of) perturbation of b in Equation (1), which makes it inconsistent. It implies that we still have least-squares solutions in $\mathrm{Im}(C)$.

A projection method that fits with the above considerations is the Kaczmarz successive projection algorithm reported in [18]. The properties (8)–(12) were proved in [30]. But independently on the general approach presented in this section, a theorem of the form Theorem 2.2 was proved in [30], the extension of the form (36)–(38) was first proposed in [24], a constrained version of the Kaczmarz method was reported in [19] and a constrained version of the Kaczmarz extended method was reported in [26]. In Section 3, we obtain all these versions for Cimmino's algorithm by simply proving that it satisfies assumptions (8)–(12) and then by applying the above general constructions and results.

3. Application – Cimmino's reflection algorithm

Cimmino [9] considered a consistent problem of the form (1), where $\mathbf{A}$ is an $m \times n$ real matrix and $b \in \mathbb{R}^m$. A solution point will lie in the intersection of the m hyperplanes described by

$$H_i := \{x \mid \mathbf{A}_i^{\mathrm{T}} x = b_i\}, \quad i = 1, \dots, m. \tag{93}$$

Given a current approximation x^k, the next one x^{k+1} is constructed as

$$x^{k+1} = \sum_{i=1}^{m} \frac{\omega_i}{\omega} y^{k,i}, \tag{94}$$

where $y^{k,i}$ are the reflections of x^k with respect to the hyperplane (93), defined by

$$y^{k,i} = x^k + 2\frac{b_i - \mathbf{A}_i^{\mathrm{T}} x^k}{\|\mathbf{A}_i\|^2}\mathbf{A}_i \quad \text{and} \quad \omega_i > 0, \quad \omega = \sum_{i=1}^{m} \omega_i. \tag{95}$$

From Equations (94) and (95), we derive for $\mathbf{T}$ and $\mathbf{R}$ in Equation (3) the following expressions:

$$\mathbf{T} = \sum_{i=1}^{m} \frac{\omega_i}{\omega}\mathbf{S}_i, \quad \mathbf{S}_i := I - 2\frac{\mathbf{A}_i\mathbf{A}_i^{\mathrm{T}}}{\|\mathbf{A}_i\|^2}, \quad \mathbf{R} = \sum_{i=1}^{m} \frac{\omega_i}{\omega}\frac{b_i}{\|\mathbf{A}_i\|^2}\mathbf{A}_i. \tag{96}$$

Then, Cimmino's algorithm (94) can be written as follows.

ALGORITHM 5 Cimmino (Cmm)
Initialization: $\omega_i > 0, i = 1, \dots, m; x^0 \in \mathbb{R}^n$.
Iterative step:

$$x^{k+1} = \mathbf{T}x^k + \mathbf{R}b. \tag{97}$$

PROPOSITION 3.1 *If*

$$\operatorname{rank}(\mathbf{A}) \geq 2, \tag{98}$$

then the matrices $\mathbf{T}$ *and* $\mathbf{R}$ *from Equation* (*96*) *satisfy assumptions* (*8*)–(*12*).

Proof The statements in Equations (8)–(11) follow directly from Equation (96) and the fact that $\mathcal{R}(\mathbf{A}^{\mathrm{T}}) = \operatorname{span}\{\mathbf{A}_1, \dots, \mathbf{A}_m\}$. For Equation (12), we first observe that the orthogonal reflectors $\mathbf{S}_i$ from Equation (96) are also isometric transformations, thus

$$\|\mathbf{S}_i x\| = \|x\|, \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \|\mathbf{S}_i\| = 1, \quad \forall i = 1, \dots, m. \tag{99}$$

Then, for an arbitrary $x \in \mathbb{R}^n$, from Equations (95) and (99), we get

$$\|\mathbf{T}x\| = \left\| \sum_{i=1}^{m} \frac{\omega_i}{\omega}\mathbf{S}_i x \right\| \leq \sum_{i=1}^{m} \frac{\omega_i}{\omega}\|\mathbf{S}_i x\| = \|x\|, \tag{100}$$

which together with Equations (9) give us $\|\mathbf{T}\| = 1$, thus

$$\|\widetilde{\mathbf{T}}\| \leq 1. \tag{101}$$

Let us now suppose that we have equality in Equation (101) and let $x \in \mathcal{R}(\mathbf{A}^{\mathrm{T}}), x \neq 0$, be such that (see also Equation (100))

$$\|\widetilde{\mathbf{T}}(x)\| = \|\mathbf{T}(x)\| = \left\| \sum_{i=1}^{m} \frac{\omega_i}{\omega}\mathbf{S}_i x \right\| = \sum_{i=1}^{m} \frac{\omega_i}{\omega}\|\mathbf{S}_i x\| = \| x \| . \tag{102}$$

By the non-singularity of $\mathbf{S}_i$, we have $\mathbf{S}_i x \neq 0$ for all $i = 1, \dots, m$. Since the Euclidean norm is strictly convex, $\omega_i > 0$ and $\sum_{i=1}^{m} \omega_i/\omega = 1$, the equality from Equation (102) only holds if

$\mathbf{S}_1x = \cdots = \mathbf{S}_mx$. Let us suppose that $\mathbf{S}_1x = \mathbf{S}_ix$ for all $i = 2, \dots, m$. This is equivalent to

$$\frac{\mathbf{A}_1^{\mathrm{T}}x}{\|\mathbf{A}_1\|^2}\mathbf{A}_1 - \frac{\mathbf{A}_i^{\mathrm{T}}x}{\|\mathbf{A}_i\|^2}\mathbf{A}_i = 0, \quad i = 2, \dots, m.$$

Since we have assumption (98) on $\mathbf{A}$, the above equalities imply that $\mathbf{A}_i^{\mathrm{T}}x = 0,\ \forall i = 1, \dots, m$, that is, $x \in \mathcal{N}(\mathbf{A})$, thus $x = 0$, which contradicts the initial assumption on it. Thus, Equation (12) holds and the proof is complete. ■

According to the results given in Section 2, we can now design the extended Cimmino algorithm following the general formulations (36)–(38). According to Equations (27) and (96), the matrix $\mathbf{U}$ from Equation (30) will be given by

$$\mathbf{U} = \sum_{i=1}^{n} \frac{\alpha_j}{\alpha}\mathbf{F}_j, \quad \text{with} \quad \mathbf{F}_j = I - 2\frac{\mathbf{A}^j{\mathbf{A}^j}^T}{\|\mathbf{A}^j\|^2}, \quad \text{and} \quad \alpha = \sum_{j=1}^{n}\alpha_j, \tag{103}$$

and $\alpha_j > 0$ arbitrary weights.

Algorithm 6 Extended Cimmino (ECmm)
Initialization: $\omega_i > 0, i = 1, \dots, m; \alpha_j > 0, j = 1, \dots, n, x^0 \in \mathbb{R}^n, y^0 = b$.
Iterative step:

$$y^{k+1} = \mathbf{U}y^k, \tag{104}$$

$$b^{k+1} = b - y^{k+1}, \tag{105}$$

$$x^{k+1} = \mathbf{T}x^k + \mathbf{R}b^{k+1}. \tag{106}$$

The corresponding constrained versions, CCmm and CECmm, are directly derived from Equations (97) and (104)–(106) following the general formulations (54) and (82)–(84), respectively.

4. Numerical experiments

In this paper, a *general extending and constraining procedure* for linear iterative methods has been considered. Cimmino's method [6,9] is such a linear iterative method, known to have a poor numerical performance. We chose to apply our general extending and constraining procedures on this algorithm, rather than on the more efficient ones [7], in order to underline the importance of extending and constraining.

To this end, we considered a widely used test problem as in [8,28]. The following numerical examples concentrate on the effect of the proposed general extending and constraining strategies on Cimmino's method. In [28], Cimmino's method was also considered, but it was compared with the DW algorithm. Moreover, the numbers of iterations differed (60 in [28], with respect to 500 or 2000 in the current work).

In our first set of experiments, we used the head phantom reported in [8] (63×63 pixel resolution with the scanning matrix with 1376 rays – that is, m, the number of rows in $\mathbf{A}$). A consistent and an inconsistent right-hand-side b were used in our reconstruction experiments, together with the following measures for the approximation errors [15]:

- x^{ex} = head phantom; $n = 63^2 = 3969$
- $x^{ex} = (x_1^{ex}, \dots, x_n^{ex})^{\mathrm{T}}; x^k = (x_1^k, \dots, x_n^k)^{\mathrm{T}}; \bar{x}^{ex} = \sum_{i=1}^{n} x_i^{ex}/n; \bar{x}^k = \sum_{i=1}^{n} x_i^k/n$

- Distance = $\sqrt{\sum_{i=1}^{n}(x_i^{ex} - x_i^k)^2 / \sum_{i=1}^{n}(x_i^{ex} - \bar{x}^{ex})^2}$
- Relative error = $\sum_{i=1}^{n} |x_i^{ex} - x_i^k| / \sum_{i=1}^{n} x_i^{ex}$
- Standard deviation = $(1/\sqrt{n})\sqrt{\sum_{i=1}^{n}(x_i^k - \bar{x}^k)^2}$
- Residual error = $\| \mathbf{A}x^k - b \|$ in the consistent case of Equation (1) and $\| \mathbf{A}^{\mathrm{T}}(\mathbf{A}x^k - b) \|$ in the inconsistent one.

In the tests, we used the box-constraining function C from Equation (51) with $\alpha_i = 0$, $\beta_i = 1$, $\forall i = 1, \dots, m$, and unitary weights ω_i and α_j in Equations (95) and (103), that is, $\omega_i = 1$, $\forall i = 1, \dots, m; \alpha_j = 1$, $\forall j = 1, \dots, n$.

Test 1: Consistent case, classical algorithms, $x^0 = 0$.

For the consistent problems associated with the head phantom, we applied the algorithm Cimmino (97) together with its constrained version (according to Equations (54) and (97)) with the initial approximation $x^0 = 0$ and 500 iterations. The results shown in Figures 1 and 2 indicate that in this case the constraining strategy used somehow improves the quality of the reconstructed image (Figure 1). According to the fact that the graphics shown in Figure 2 are almost identical, an explanation would be related to the small changes (in the positive sense) in the components of the approximations x^k, such that, also by starting with $x^0 = 0$, in 500 iterations, Cimmino's algorithm together with its constrained version acts almost identical.

Test 2: Consistent case, classical algorithms, $x_i^0 = (-1)^i, i = 1, \dots, n$.

We performed tests similar to Test 1, but with the initial approximation $x_i^0 = (-1)^i, i = 1, \dots, n$. Figure 3 shows that the constraining strategy used much improved the quality of the reconstructed image in this case. This aspect can also be seen in the graphics shown in Figure 4.

Remark 9 In real reconstruction problems, we never use an initial approximation as in Test 2. The idea in these experiments was to show that the constraining strategy can be a very powerful tool in improving the quality of the reconstruction. The real solution for Test 1 would be an adaptive constraining strategy. Work is in progress on this subject.

Test 3: Inconsistent case, combined algorithms.

For the inconsistent problems associated with both phantoms, we applied the algorithm Cimmino (97) together with the extended Cimmino algorithms (104)–(106), 500 iterations and $x^0 = 0$. The results shown in Figures 5 and 6 indicate better results for the classical version (97). Although strange, this behaviour can be explained by the fact that the better theoretical properties of the extended Cimmino algorithms (104)–(106), as derived in Theorem 2.6, have an 'asymptotic' nature. More clearly, this means that they become 'visible' after a consistently large number of iterations have been achieved (see in this sense, similar experiments presented in Figures 7 and

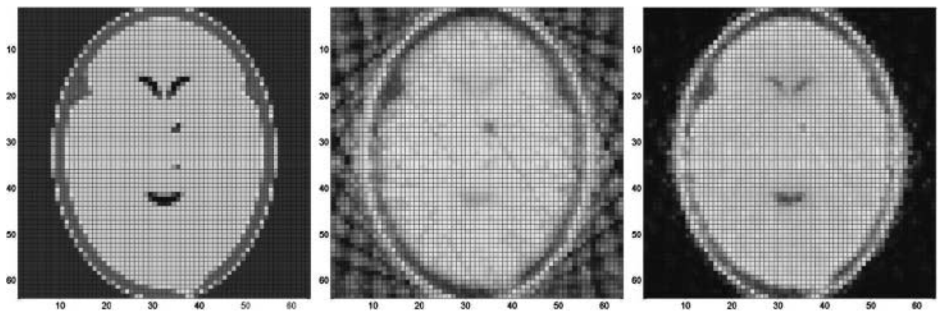

Figure 1. Consistent case, $x^0 = 0$, 500 iterations; left, exact; middle, Cmm; right, CCmm.

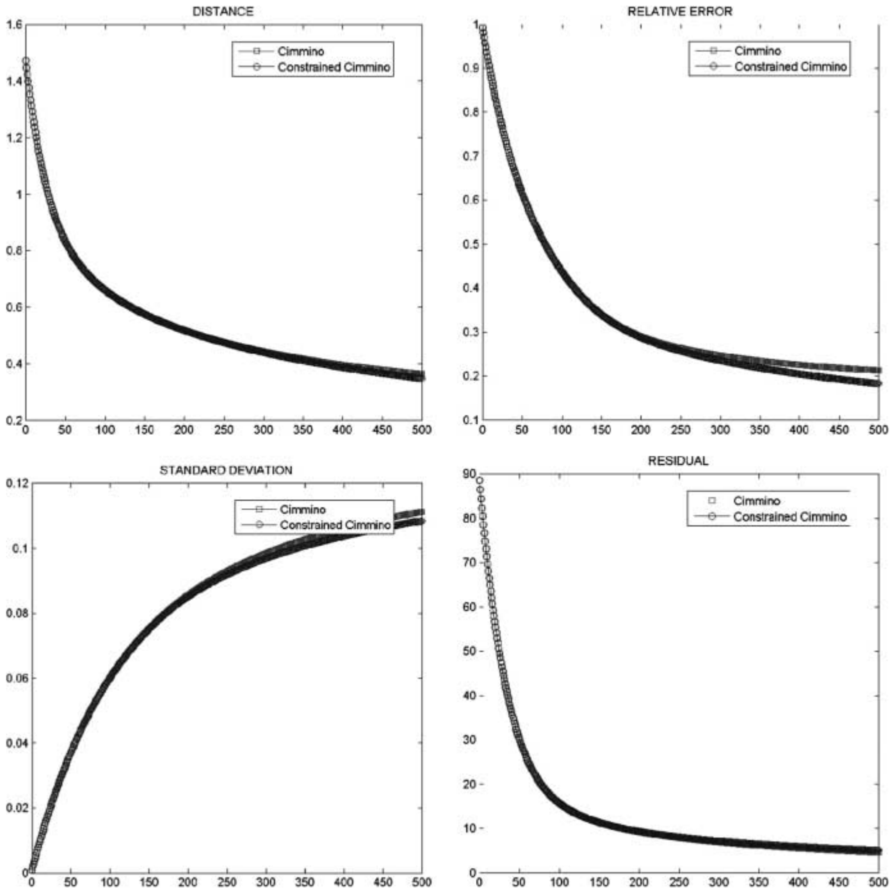

Figure 2. Consistent case, $x^0 = 0$, 500 iterations; errors.

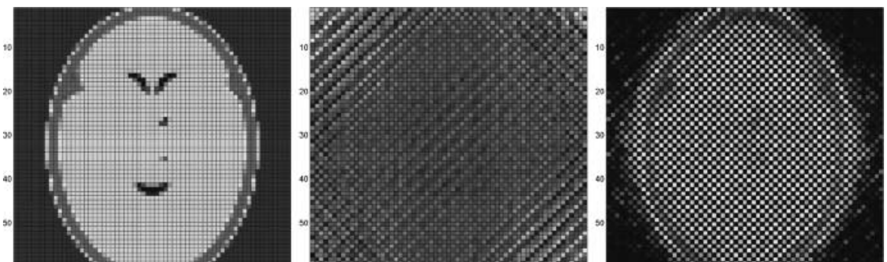

Figure 3. Consistent case, $x_i^0 = (-1)^i, i = 1, \ldots, n$, 500 iterations; left, exact; middle, Cmm; right, CCmm.

8, for which 2000 iterations were used). A solution of this problem would be to improve the 'right-hand-side correction part' (104). Some steps have been already taken in this direction [27].

In our second set of experiments, we considered the problem of 3D particle image reconstruction, which is the main step of a new technique for imaging turbulent fluids, called *TomoPIV* [13]. This technique is based on the instantaneous reconstruction of particle volume functions from few and simultaneous projections (2D images) of the tracer particles within the fluid. TomoPIV adopts a simple discretized model for an image-reconstruction problem, which assumes that the image

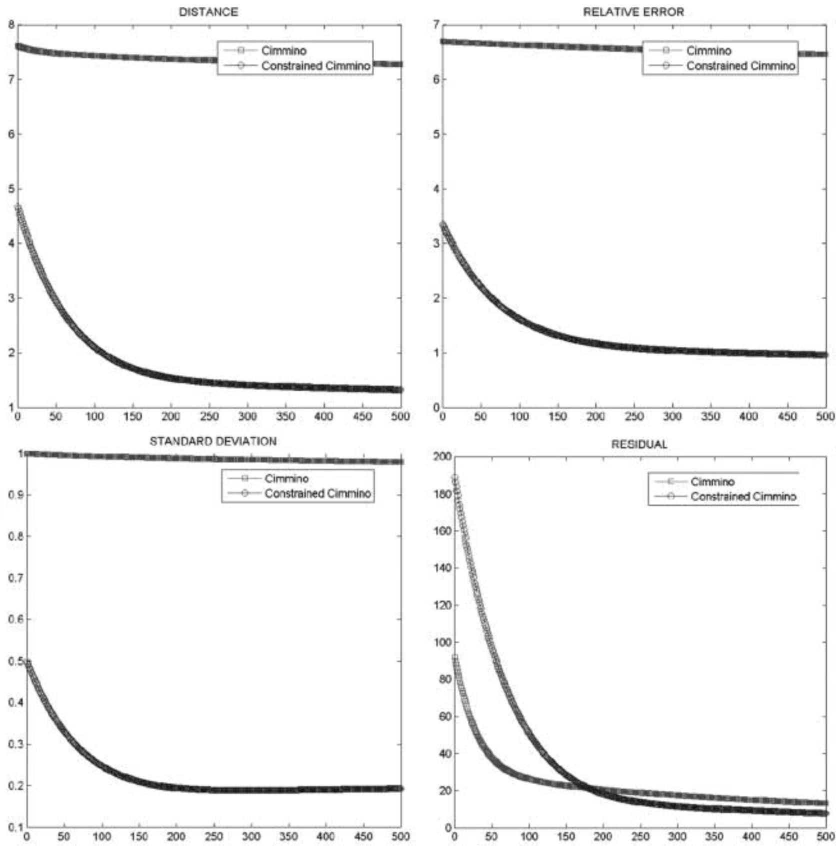

Figure 4. Consistent case, $x_i^0 = (-1)^i, i = 1, \ldots, n$, 500 iterations; errors.

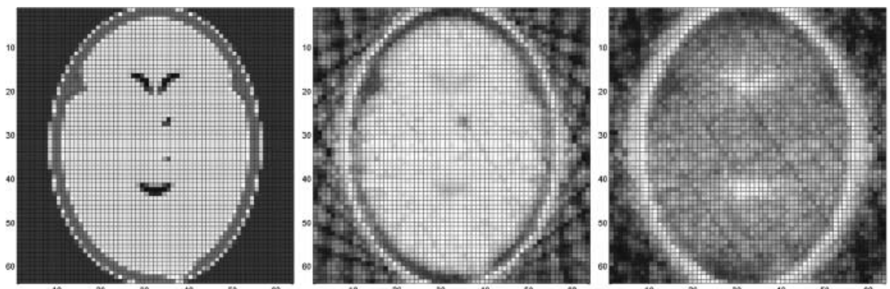

Figure 5. Inconsistent case, $x^0 = 0$, 500 iterations; left, exact; middle, Cmm; right, ECmm.

consists of an array of unknowns (voxels) and sets up algebraic equations for the unknowns in terms of the measured projection data. The latter are the pixel entries in the recorded 2D images. TomoPIV employs undersampling due to the cost and complexity of the measurement apparatus, resulting in an underdetermined system of equations and thus in an ill-posed image-reconstruction problem. However, this reconstruction problem can be modelled as finding the sparsest solution of an underdetermined linear system of equations, that is,

$$\min \|x\|_0 \quad \text{such that } \mathbf{A}x = b, \tag{107}$$

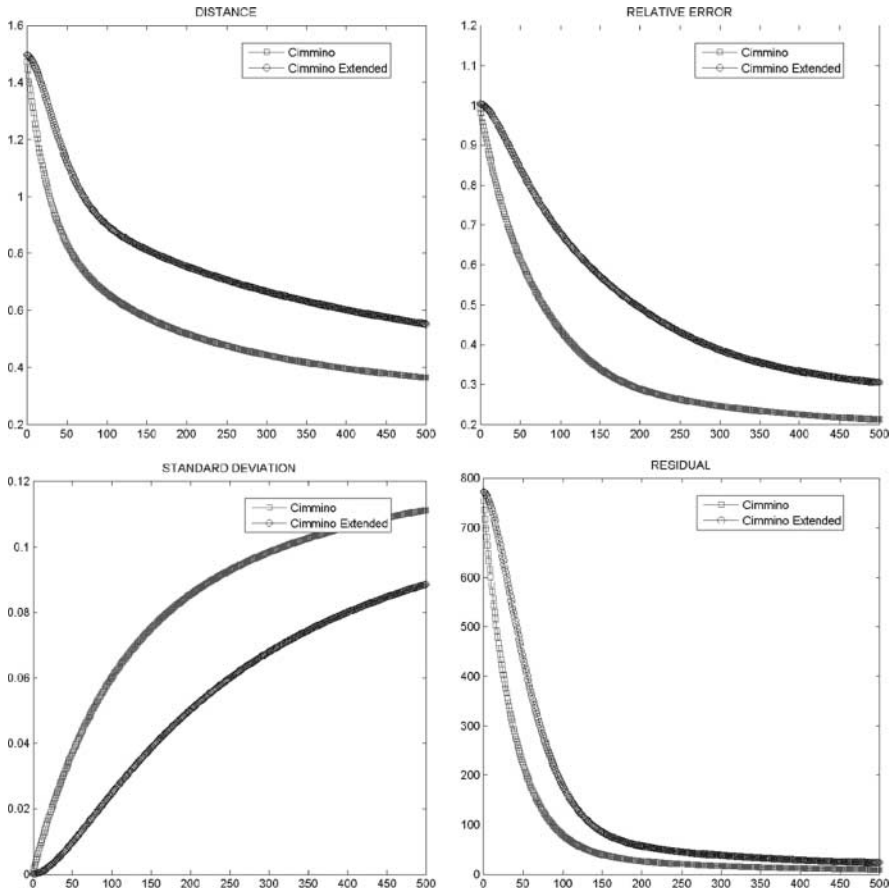

Figure 6. Inconsistent case, $x^0 = 0$, 500 iterations; errors.

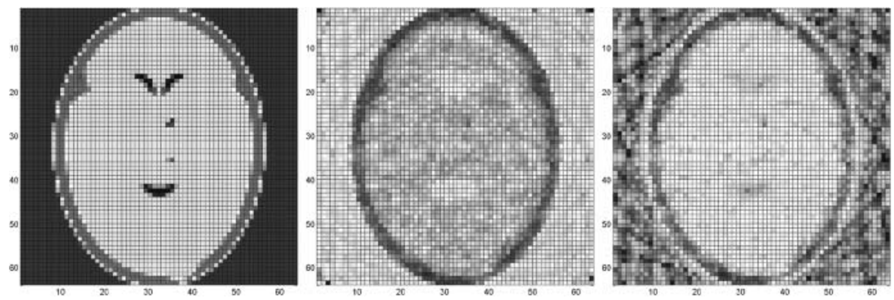

Figure 7. Inconsistent case, $x^0 = 0$, 2000 iterations; left, exact; middle, Cmm; right, ECmm.

since the original particle distribution can be well approximated with only a very small number of active voxels relative to the number of possible particle positions in a 3D domain; see, for example, [22] for details. If the original particle distribution is sparse enough and the coefficient matrix satisfies certain properties, then the indicator vector (corresponding to the active voxels) is also the unique non-negative vector which satisfies the measurements $\mathbf{A}x = b$ and coincides with the solution of Equation (107).

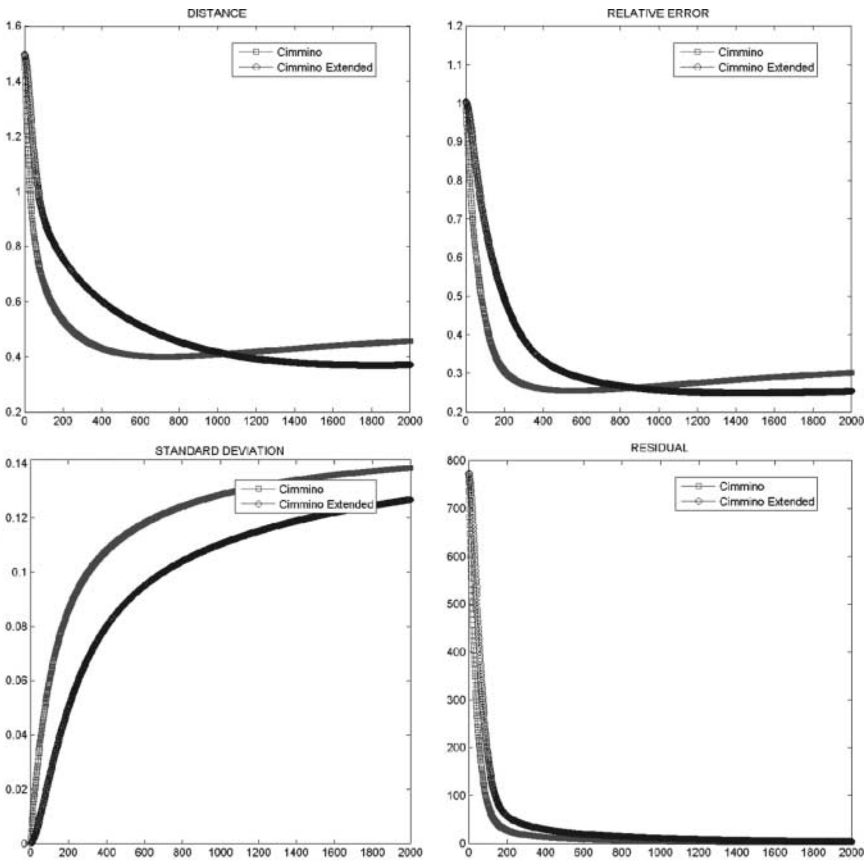

Figure 8. Inconsistent case, $x^0 = 0$, 2000 iterations; errors.

Here, we concentrate on a simple geometry for sampling the original particle distribution within a $64 \times 64 \times 64$ 3D domain from three orthogonal directions (Figure 9). The sampling matrix $\mathbf{A}$ will correspond to a perturbed adjacency matrix of a bipartite graph [22] where the left nodes correspond to the 64^3 voxels within the volume and the right nodes to the $3 \cdot 64^2$ pixels within the three sampled 2D images (Figure 9(right)).

If the number of non-zero elements in the original indicator vector x^*, that is, $\|x^*\|_0 := |\{i|x_i^* \neq 0\}|$, is small enough, more precisely

$$q := \|x^*\|_0 \leq \frac{3 \cdot 64^2}{4\log(64/3)} \approx 1003,$$

then x^* is (most probably) the unique *non-negative* solution of the linear system $\mathbf{A}x = b$. Although the hard thresholding constraining function does not satisfy all assumptions (48)–(50) (Example 2.8), we used it in combination with Cimmino's algorithm because Cimmino's algorithm combined with this constraining strategy is closely related to the hard thresholded Landweber iteration reported in [4], where this method is shown to converge to a local optimum of Equation (107). Due to this attribute, the method was only applied in [4] as a preprocessing step for the solution refinement obtained by other sparse approximation algorithms. However, for a carefully chosen threshold α, the method will converge to the solution of Equation (107), provided that this sparsest solution is unique. But choosing the proper α is an art by itself. We decided to use

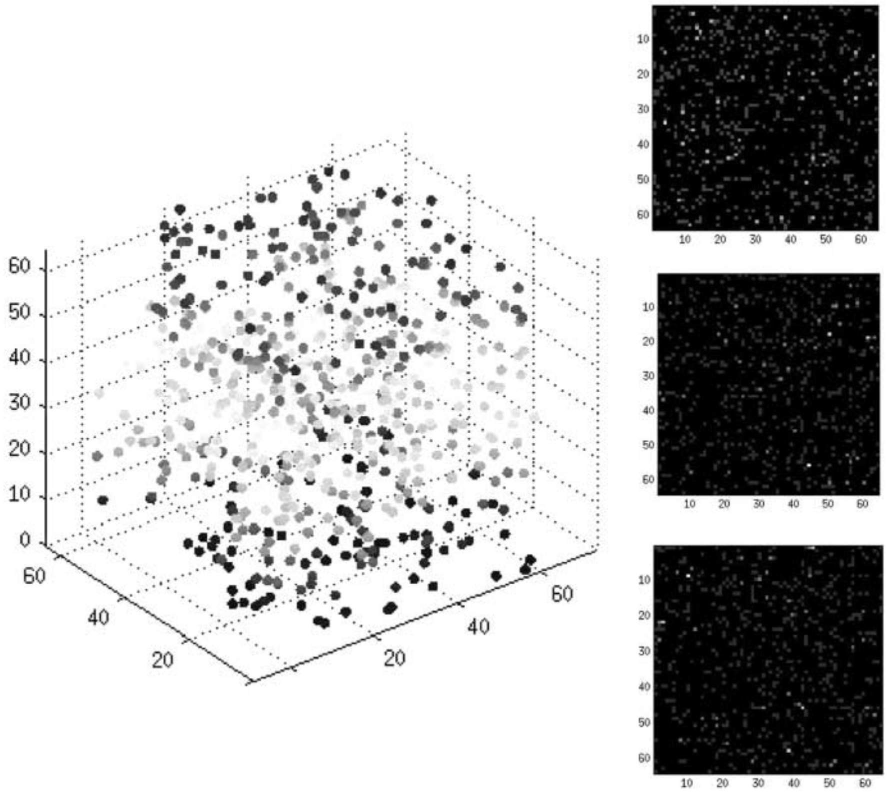

Figure 9. The original 3D particle volume function (e.g. 602 particles) that has to be reconstructed from three 2D images (64^2 pixel each). The indicator vector corresponding to the original particle distribution is also the unique non-negative solution which satisfies the measurements.

$\alpha = 0.1$ but only after the first $0.5\,q$ iterations. A combination of two constraining operators is also legitimate and turns out to be more effective in reducing the error within the same number of iterations (Figure 10). In fact, hard thresholding combined with box constraining has an acceleration effect reflected in a reduced number of iterations (Figure 11). Hence, a proper constraining strategy is an indispensable tool not only for regularization purposes (box constraining) but also for achieving computational efficiency. This issue will be addressed in future work.

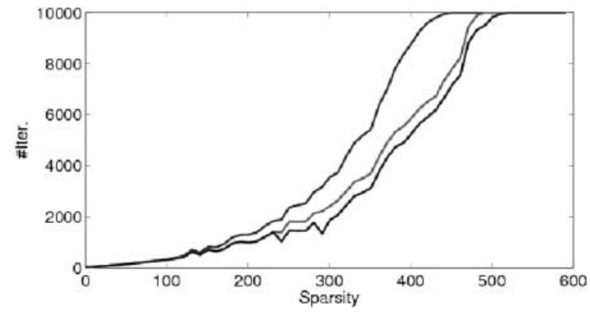

Figure 10. Reconstruction experiment for 10, 20, ..., 600 particles (sparsity) within the 64^3 cube for Cimmino's algorithm with box constraining (left), Cimmino's algorithm with hard thresholding (middle) and Cimmino's algorithm combined with both constraining strategies (right). The averaged number of iterations (100 trials) increases proportionally to the particle sparsity. Thresholding turns out to be more effective in attaining the stopping rule: relative error less than 10^{-2} or more than 10,000 iterations.

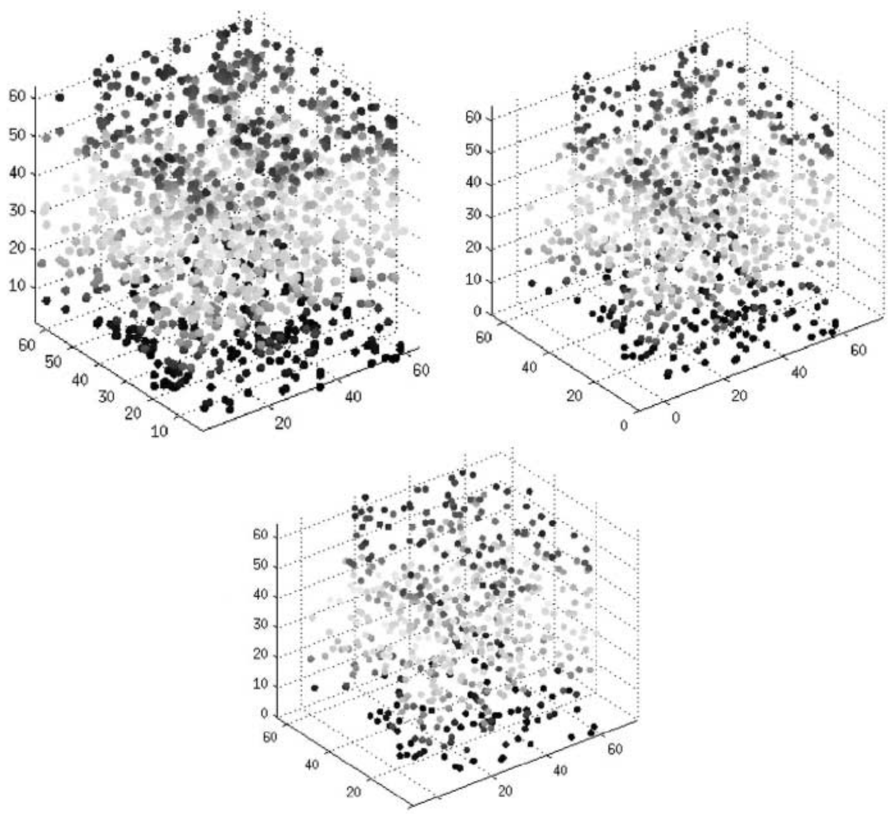

Figure 11. Reconstruction experiment for 602 particles in a $64 \times 64 \times 64$ cube from three orthogonal projections. In 1000 iterations of Cimmino's algorithm with box constraining (51), the reconstruction (top left) contains 1246 particles exceeding a threshold 0.5. By combining box constraining with the hard thresholding operator from Equations (52) and (53), the reconstruction improves. After 1000 iterations, the reconstruction (top left) contains 827 particles – containing the original ones. The reconstruction after convergence is exact, that is, identical to the original shown in Figure 9 within the tolerance – box constraining: 18,029 iterations and box constraining combined with hard thresholding: 30,787 iterations (bottom).

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